Assignment 3

Modern Theory of Markov Chains

Due: 05.03.2013

1 (Markov property)**.** Which of the following sequences is a Markov chain? Prove your claims. For each that is a Markov chain, identify the transition probabilities.

a) The sequence $|W_1|, |W_2|, \ldots$, where $W_1, W_2, \ldots$ is a simple random walk on $\mathbb{Z}$.

b) A simple random walk $R_1, R_2, \ldots$ on $\mathbb{Z}^2$ with the additional restriction that the walk cannot pass any position more than once.

c) The sequence $X_1, X_2, \ldots$ where $X_i \triangleq |\{Z_1, Z_2, \ldots, Z_i\}|$ and $Z_1, Z_2, \ldots,$ are independent uniformly distributed random variables in $\{1, 2, \ldots, n\}$.

2 (Markov property)**.** The Markov property is often put in words as

> *"Given the present, the past is irrelevant for predicting the future."*

In the mathematical formulation, a sequence of random variables

$$X_0, X_1, X_2, \ldots$$

is said to satisfy the Markov property if for every $t \geq 0$,

$$\mathbf{P}(X_{t+1} = a_{t+1} \mid X_0 = a_0, X_1 = a_1, \ldots, X_t = a_t) = \mathbf{P}(X_{t+1} = a_{t+1} \mid X_t = a_t) ,$$

or more concisely

$$\mathbf{P}(X_{t+1} = a_{t+1} \mid X_0, X_1, \ldots, X_t) = \mathbf{P}(X_{t+1} = a_{t+1} \mid X_t) . \tag{M}$$

This might seem as a rather restricted formulation of the latter intuitive description. A more satisfying formulation would be

$$\mathbf{P}(E_{\text{future}} \mid X_t = a \wedge E_{\text{past}}) = \mathbf{P}(E_{\text{future}} \mid X_t = a) , \tag{M*}$$

where E_{future} is any "event of interest" depending on $X_{t+1}, X_{t+2}, \ldots$ and E_{past} is any "event of interest" depending on $X_0, X_1, \ldots, X_t$.

Suppose that a sequence $X_0, X_1, X_2, \ldots$ with values from a countable set S satisfies property (M).

a) Verify that for every $t \geq 0$ and every random variable Y_t which is a function of $X_0, X_1, \ldots, X_t$,

$$\mathbf{P}(X_{t+1} = a_{t+1} \mid X_t, Y_t) = \mathbf{P}(X_{t+1} = a_{t+1} \mid X_t) .$$

b) Verify that for every $t \geq 0$ and $k \geq 0$ and every event E that depends only on $X_{t+1}, X_{t+2}, \ldots, X_{t+k}$,

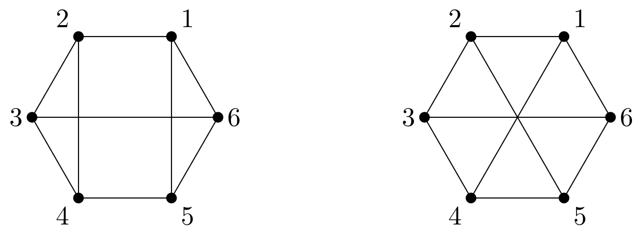
$$\mathbf{P}(E \mid X_0, X_1, \ldots, X_t) = \mathbf{P}(E \mid X_t)$$

c) Let T_a^t be the smallest time $s > t$ such that $X_s = a$ (set $T_a^t = \infty$ if such a time s does not exist). Verify that for every $n \in \mathbb{N}$,

$$\mathbf{P}\{T_a^t = n \,|\, X_0, X_1, \ldots, X_t) = \mathbf{P}(T_a^t = n \,|\, X_t) \ .$$

A complete proof of (M*) in its strongest generality might require a bit of measure theory. With the help of measure theory, (M*) can be proved for all events E_{future} that can be obtained by means of countable unions, countable intersections, and complementations from the events $\{X_s = a_s\}$ for $s > t$, and all events E_{past} obtained similarly from $\{X_s = a_s\}$ for $s \leq t$.

3 (Asymptotic behaviour)**.** Consider a simple random walk on each of the following graphs.



G_1 G_2

a) Using a pseudorandom number generator, make three independent realizations of the walk on G_1 from time 0 to 20 starting from vertex 1, and plot the results as a function of time. Repeat this for the walk on G_2.

b) Now generate 200 independent realizations of the walk on G_1 from time 0 to 50 starting from vertex 1. For each time step $0 \leq t \leq 50$, calculate the empirical distribution of the position of these 200 walks, and plot this empirical distribution as a function of time. (If you are using Mathematica, you can, for example, use the `ArrayPlot` function.) Repeat this for the walk on G_2. How do you explain the qualitative difference between the results of the experiments on G_1 and G_2?

c) Generate a single realization of the walk on G_1 from time 0 to $N = 700$, starting from vertex 1. For each time $0 \leq t \leq N$, calculate the density of the time steps $0 \leq s \leq t$ in which the walk has been in position 1 (or $2, 3, \ldots, 6$). Plot this density as a function of time t. Does this density seem to converge to a constant? If necessary, vary the length of the experiment N to see the trend. Repeat the above for the walk on G_2.

d) Pick your favorite function $f : \{1, 2, \ldots, 6\} \to \mathbb{R}$. As before, generate a single realization of the walk on either G_1 or G_2 from time 0 to $N = 700$, starting from vertex 1. For each time $0 \leq t \leq N$, calculate the average value of f over the position of the walk in times 0 up to t. Plot this average value as a function of time t. Does this average value seem to converge to a constant? Vary N to see the trend.

e) Can you explain the observation in part (d) using the observation in part (c)?