Assignment 2

Modern Theory of Markov Chains

Due: 19.02.2013

1 (Conditional expectations; Wald's equation)**.**

a) Let $Z_1, Z_2, \ldots$ be independent copies of a random variable Z. Set $S_n \triangleq Z_1 + \cdots + Z_n$ for $n \geq 1$ and $S_0 \triangleq 0$. Let N be another random variable with values from $\mathbb{N} = \{0, 1, 2, \ldots\}$ and *not necessarily* independent of $Z_1, Z_2, \ldots$. Assuming that Z and N both have finite expected values, show that $\mathbf{E}\, S_N = \mathbf{E}\, Z\, \mathbf{E}\, N$.

b) Consider a particle having a simple random walk on the discrete line $\mathbb{Z}$, starting from the origin. Let k and l be two positive integers, and let T be the first time that the particle leaves the interval $(-k, l)$. What is the expected value of the position of the particle at time T? What is the probability that the particle reaches point l first?

2 (Ehrenfest urn)**.** Consider the Ehrenfest urn with n balls.

a) (Exercise 2.5 of the textbook) Show that the binomial distribution with parameters n and $1/2$ is a stationary distribution.

(BONUS) b) Show that the model has no other stationary distribution?

c) Does the distribution of the chain converge to the stationary distribution as time goes to infinity?

3 (Random walks)**.**

a) What are the stationary distributions of a simple random walk on $\mathbb{Z}$?

b) What are the stationary distributions of a random walk on $\mathbb{Z}$ that is biased towards right with parameter $0 < p < 1$? (That is, at each step, the particle moves to the right with probability p and to the left with probability $1 - p$.)

c) What are the stationary distributions of a biased random walk on a cycle of length n with parameter $0 < p < 1$?

(BONUS) **4** (A paradox of probabilities)**.** Suppose that we flip a fair coin over and over till eternity. The sample space of this experiment is the set of all infinite sequences of heads and tails

$$\Omega \triangleq \{\omega_1\omega_2\cdots : \forall i, \omega_i \in \{\mathtt{H}, \mathtt{T}\}\} = \{\mathtt{H}, \mathtt{T}\}^{\mathbb{Z}^+} .$$

The fairness of the coin suggests that the probability of getting a head in each flip is $1/2$, and the implicit assumption is that the outcome of different flips are independent. These for instance imply that the probability the event that the 5th, 6th and 7th flips turn out `H`, `H`, and `T` respectively is

$$\begin{aligned}\mathbf{P}(\{\omega : \omega_5\omega_6\omega_7 = \mathtt{HHT}\}) &= \mathbf{P}(\{\omega : \omega_5 = \mathtt{H}\}) \cdot \mathbf{P}(\{\omega : \omega_6 = \mathtt{H}\}) \cdot \mathbf{P}(\{\omega : \omega_7 = \mathtt{T}\}) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} .\end{aligned}$$

So far, so good! Consider now a relation $\sim$ on Ω defined as follows: $\omega \sim \omega'$ if and only if the sequences $\omega_1\omega_2\cdots$ and $\omega'_1\omega'_2\cdots$ *eventually agree*, meaning that there is an index $n \in \mathbb{Z}^+$ such that $\omega_i = \omega'_i$ for all $i \geq n$.

a) Verify that $\sim$ is an equivalence relation.

Let W be a maximal set such that no two elements of W are related by $\sim$. Then W has exactly one element from each equivalence class of $\sim$. For every finite set of indices $I \subseteq \mathbb{Z}^+$, we can obtain a new set W^I from W by switching the value of the elements indexed by I in every sequence $\omega \in W$. For example, if $I = \{1, 2, 4\}$ and W contains the sequence $\texttt{TTTTTTTTT}\cdots$, then W^I must contain the sequence $\texttt{HHTHTTTTT}\cdots$ and vice versa. As another example, $W^{\varnothing}$ is the same as W itself.

b) Argue for or against: the sets W^I must all have the same probability.

c) Prove that the sets W^I are disjoint and that the union of W^I is the whole space Ω.

d) Verify that the family $\{W^I : I \subseteq \mathbb{Z}^+ \text{ finite}\}$ is countable.

e) What is the probability that the outcome sequence of the coin flips falls in W? Do you see any inconsistency? If yes, what would be your approach to resolve the paradoxical situation?