Assignment 1

Modern Theory of Markov Chains

Due: 19.02.2013

1 (Exercise 2.1 in the textbook)**.** A drunkard is wandering aimlessly on a street that has n blocks. Starting from the kth corner ($0 < k < n$), he randomly aims towards one of the two directions along the street and walks until he reaches another corner, and he repeats this process until he reaches either the 0th corner (his home) or the nth corner (the pub), where he spends the rest of the night. Show that the expected number of stages it takes for him to reach his final position is $k(n-k)$.

2 (Coin flips)**.** Suppose that we flip a fair coin over and over (infinitely many times).

a) What is the probability that the pattern HTHH never appears in four consecutive flips?

b) What is the probability that only finitely many heads appear?

3 (Continuity of the probabilities)**.** In elementary probability theory, the sample space (the space of all possible outcomes) — let us denote it by Ω — is typically finite or at most countably infinite. In such a situation, one can identify the model simply by specifying the probabilities of the occurrence of each outcome $\omega \in \Omega$, that is, $\mathbf{P}(\omega)$. The probability of an event is then the sum of the probabilities of all outcomes that realize that event. If the sample space of the model is not countable (for example, if we want to talk about a uniform random variable in the interval $[0, 1]$), the approach of assigning a probability to each possible outcome is not sufficient.

a) How would you define the probability of an event regarding a number chosen uniformly at random from the interval $[0, 1]$? What is the probability that the number falls in the set $(1/4, 1/2)$? What is the probability of an individual outcome $\omega \in [0, 1]$?

Whatever the definition of the probabilities, there are at least three constraints that obviously should be satisfied in order to have a consistent model:

- *Valid values*: the probability of each event E is a real number $\mathbf{P}(E)$ between 0 and 1,
- *Exhaustion of possibilities*: $\mathbf{P}(\Omega) = 1$,
- *Additivity*: $\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F)$ for every two *disjoint* events E and F.

Many other properties follow logically from these three.

b) What is the probability of the empty event $\varnothing$? Present an interpretation!

c) Prove that the probabilities are *sub-additive*: $\mathbf{P}(E \cup F) \leq \mathbf{P}(E) + \mathbf{P}(F)$ for every two events E and F (not necessarily disjoint). Present an interpretation!

Aside from the above constraints, it is also intuitively plausible (and technically crucial) that the probabilities are also (monotonically) *continuous*, in the sense that

- ▷ $\mathbf{P}(\lim_{i\to\infty} E_i) = \lim_{i\to\infty} \mathbf{P}(E_i)$ for every sequence of events $E_1, E_2, \ldots$ that is either increasing $E_1 \subseteq E_2 \subseteq \cdots$ or decreasing $E_1 \supseteq E_2 \supseteq \cdots$.

(You might have already used this in solving Problem 2!)

d) Show that additivity, exhaustion of possibilities, and monotone continuity imply *countable additivity*: $\mathbf{P}(E_1 \cup E_2 \cup \cdots) = \mathbf{P}(E_1) + \mathbf{P}(E_2) + \cdots$ for any sequence of *disjoint* events $E_1, E_2, \ldots$.

e) Conversely, show that countable additivity and exhaustion of possibilities imply monotone continuity.