American University of Beirut

MATH/STAT 233: Probability

2022–2023 Spring

Assignment 7 (due: Wednesday, April 26)

Problem 1 (Cauchy–Schwarz inquality).

(a) Prove that for every two random variables X and Y,

$$(\mathbb{E}[XY])^2 \leq \mathbb{E}[X^2]\,\mathbb{E}[Y^2] \tag{✐}$$

provided the expectations exist.
[*Hint*: Observe that the function $\varphi(t) \coloneqq \mathbb{E}[(Y - tX)^2]$ is non-negative for every $t \in \mathbb{R}$. Expand the expression for $\varphi(t)$ as a quadratic polynomial of a single variable t. What does the non-negativity of $\varphi(t)$ tell you about the coefficients of the latter polynomial?]

(b) Argue that the equality in (✐) holds if and only if either $X = 0$ or $Y = cX$ for some $c \in \mathbb{R}$. Describe the latter condition in a symmetric form.

Problem 2 (Covariance and correlation). Recall that the *covariance* of two random variables X and Y is defined as $\mathbb{C}\mathrm{ov}[X, Y] \coloneqq \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right]$ when the expectation exists.

(a) Verify that if X and Y are independent, then $\mathbb{C}\mathrm{ov}[X, Y] = 0$.

(b) Verify that the converse of the latter statement is true if X and Y are Bernoulli random variables.

(c) Find an example of a pair of random variables X and Y with $\mathbb{C}\mathrm{ov}[X, Y] = 0$ that are not independent.
[*Hint*: To simplify things, look for an example with $\mathbb{E}[X] = \mathbb{E}[Y] = 0$. Tune the joint distribution of X and Y in such a way that always $XY = 0$.]

The *correlation coefficient* of X and Y is defined as

$$\rho(X, Y) \coloneqq \frac{\mathbb{C}\mathrm{ov}[X, Y]}{\sqrt{\mathbb{V}\mathrm{ar}[X]\,\mathbb{V}\mathrm{ar}[Y]}} \ ,$$

provided the variances are non-zero.

(d) Verify that $-1 \leq \rho(X, Y) \leq 1$.
[*Hint*: Apply the Cauchy–Schwarz inequality from Problem 1 to the centered versions of X and Y.]

(e) Find examples of pairs of random variables with correlation coefficients 1 and -1.
[*Hint*: These are the cases in which the Cauchy–Schwarz inequality is actually an equality.]

Problem 3 (Minimum and maximum of independent exponential RVs). Let $X \sim \mathsf{Exp}(\lambda)$ and $Y \sim \mathsf{Exp}(\gamma)$ be independent exponential random variables with rates λ and γ respectively. Let $V \coloneqq \min\{X, Y\}$ and $W \coloneqq \max\{X, Y\}$.

(a) Find the distribution of W.
[*Hint*: Start with the cdf of W. Use the fact that $\max\{a, b\} \leq w$ if and only if both $a \leq w$ and $b \leq w$.]

(b) Find the distribution of V. Does the result surprise you?
[*Hint*: Start with the cdf of V. Use the fact that $\min\{a, b\} > v$ if and only if both $a > v$ and $b > v$.]

(c) Find the joint distribution of V and W.
[*Hint*: Start with the joint cdf of V and W. Given $v, w \in \mathbb{R}$, identify the region of the x-y plane in which $\min\{x, y\} \leq v$ and $\max\{x, y\} \leq w$. Use inclusion-exclusion to write the probability of this region in terms of the joint cdf of X and Y.]

Problem 4 (Bernoulli RVs). Let X and Y be a pair of Bernoulli random variables with joint distribution

$$p(0,0) = a \ , \qquad p(1,0) = b \ , \qquad p(0,1) = c \ , \qquad p(1,1) = d \ .$$

Show that

$$\mathbb{E}[X \mid Y] = \frac{b}{a+b} + \frac{ad - bc}{(a+b)(c+d)} Y \ .$$

Problem 5 (Exponential with random parameter). Let W be a positive random variable, and suppose that, given $W = w$, the conditional distribution of X is exponential with rate w.

(a) Assuming $W \sim \mathsf{Geom}(p)$, find the probability density function of X.

(b) Assuming $W \sim \mathsf{Unif}([0,1])$, find the probability density function of X.

[*Hint*: In each case, start with finding the cumulative distribution function of X. Use the law of total probability $\mathbb{P}(X \leq x) = \mathbb{E}[\mathbb{P}(X \leq x \mid W)]$.]

Problem 6 (Drunkard's walk). Consider a drunkard performing a random walk down a street, moving at each step either one step forward or one step backward, with probabilities p and $1-p$ respectively. He stops and settles for the night if he either arrives at home (located at position 0, the west end of the street) or at the town's pub (located n steps from his home towards the east). The drunkard starts his random walk at position a, where $0 \leq a \leq n$. Let T denote the random number of steps he takes until he arrives at home or the pub. Find the expected value of T as a function of a.

More specifically, let $Z_1, Z_2, \ldots$ be i.i.d. random variables taking values $+1$ and -1, representing the consecutive steps of the drunkard. Let X_0 denote the drunkard's starting position, and for $t \geq 1$, let $X_t := X_{t-1} + Z_t$. In this setting, $T := \min\{t : X_t \in \{0, n\}\}$. Let $g(a) := \mathbb{E}[T \mid X_0 = a]$.

(a) Use conditioning on Z_1 to show that g satisfies the recursion $g(a) = 1 + pg(a+1) + (1-p)g(a-1)$ for $a = 1, 2, \ldots, n-1$, with boundary conditions $g(0) = g(n) = 0$. Observe that the latter recursion can also be written as $p\big[g(a+1) - g(a)\big] = (1-p)\big[g(a) - g(a-1)\big] - 1$.

(b) Case $p = 1/2$: Solve the recursion for g to show that $g(a) = a(n-a)$ for $a = 0, 1, 2, \ldots, n$.
[*Hint*: Rewrite the recursion in terms of $h(a) := g(a) - g(a-1)$ and solve it to find $h(a) = h(1) - 2(a-1)$ for $a = 1, 2, \ldots, n$. Then, observe that $g(a) = \sum_{k=1}^{a} h(k) + g(0)$. Lastly, use the boundary conditions.]

(c) Case $p \neq 1/2$: Solve the recursion for g to show that $g(a) = \frac{n}{2p-1}\frac{1-(q/p)^a}{1-(q/p)^n} - \frac{a}{2p-1}$ for $a = 0, 1, 2, \ldots, n$, where $q := 1-p$. [*Hint*: Find a constant β such that the recursion can be written as $p\big[g(a+1) - g(a) + \beta\big] = (1-p)\big[g(a) - g(a-1) + \beta\big]$. Solve the recursion for $h(a) := g(a) - g(a-1) + \beta$ in terms of $h(1)$. Use a telescopic sum (as in the previous part) to write $g(a)$ in terms of $h(1)$. Finally, use the boundary conditions.]