American University of Beirut

MATH/STAT 233: Probability

2022–2023 Spring

Assignment 6 (due: Wednesday, April 19)

General formulation of expected value. Recall that we defined the expected value of an arbitrary random variable X (not necessarily discrete or continuous) as follows:

- In case X is non-negative, we define $\mathbb{E}[X] \coloneqq \int_0^\infty \mathbb{P}(X > x)\,\mathrm{d}x$, provided the integral converges.
- For arbitrary X, we write $X = X^+ - X^-$ where $X^+ \coloneqq \max\{X, 0\}$ and $X^- \coloneqq \max\{-X, 0\}$ are the positive and negative parts of X respectively, and define $\mathbb{E}[X] \coloneqq \mathbb{E}[X^+] - \mathbb{E}[X^-]$.

As you proved in Problem 6 of Assignment 4, in the two cases of discrete and continuous random variables, the above definition is equivalent to the usual definitions of the expected value.

Suggested (ungraded) warm-up exercises:

- Problems 9.1, 9.2, 9.6, 9.24, 9.38, 10.4, 10.7, 10.15, 10.35, 11.6 of Tijms's book

Problem 1 (Random permutation of Mississippi)**.** Problem 9.14 of Tijms's book.

Problem 2 (Square root of an exponential RV)**.** Problem 10.29 of Tijms's book.

Problem 3 (Three random numbers)**.** Problem 11.3 of Tijms's book.

Problem 4 (Sum of independent uniform RVs)**.** Let X and Y be two independent continuous random variables, each uniformly distributed over the interval $[0, 1]$. Find the probability density function of $Z \coloneqq X + Y$ and plot its graph. [*Hint*: Use the convolution formula. Pay attention to the fact that the pdfs of X and Y have piecewise definitions, so you may need to consider different cases separately.]

Problem 5 (Ratio of independent geometric/exponential RVs)**.**

(a) Let $N_1 \sim \mathsf{Geom}(p)$ and $N_2 \sim \mathsf{Geom}(p)$ be two independent geometric random variables with parameter p. Find the distribution of $M \coloneqq N_1/N_2$. [*Hint*: Start by identifying the possible values of M.]

(b) Let $T_1 \sim \mathsf{Exp}(\lambda)$ and $T_2 \sim \mathsf{Exp}(\lambda)$ be two independent exponential random variables with rate λ. Find the distribution of $S \coloneqq T_1/T_2$. [*Hint*: Start with the cdf of S. Recall that the joint pdf of independent RVs is the product of their marginal pdfs.]

Problem 6 (Joint cdf)**.** Let X and Y be two random variables with joint cumulative distribution function $F : \mathbb{R} \times \mathbb{R} \to [0, 1]$. (Recall: $F(x, y) \coloneqq \mathbb{P}(X \leq x, Y \leq y)$.) Consider $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $a_1 < a_2$ and $b_1 < b_2$.

(a) Express the probability $\mathbb{P}(a_1 < X \leq a_2 \text{ and } b_1 < Y \leq b_2)$ in terms of F.
[*Hint*: The set $\{(x, y) : a_1 < x \leq a_2 \text{ and } b_1 < y \leq b_2\}$ represents a semi-open rectangle on the plane.]

(b) Express the probability $\mathbb{P}(a_1 \leq X \leq a_2 \text{ and } b_1 \leq Y \leq b_2)$ in terms of F.
[*Hint*: Use the result of the previous part and the continuity of probabilities as in the one-dimensional case.]

Problem 7 (Linearity of expectation)**.**

(a) Let X and Y be two jointly continuous random variables (i.e., assume they have a joint density function). Prove that $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.
[*Hint*: Follow the pattern of the proof of the discrete case discussed in class.]

(b) Prove that for every random variable X (not necessarily discrete or continuous) and every two numbers $a, b \in \mathbb{R}$, $\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b$. For simplicity, you can assume that X, a and b are all non-negative.

Problem 8 (Mixtures of distributions)**.** Let X and Y be random variables, and let B be a Bernoulli random variable with parameter $\alpha \in [0, 1]$ that is independent of X and Y. Define

$$Z \coloneqq \begin{cases} X & \text{if } B = 1, \\ Y & \text{if } B = 0. \end{cases}$$

The distribution of Z is said to be a *mixture* of the distributions of X and Y.

(a) Write the cdf of Z in terms of α, the cdf of X, and the cdf of Y.

(b) Describe the distribution of Z if (i) $X \sim \mathsf{Geom}(p)$ and $Y \sim \mathsf{Geom}(q)$. (ii) $X \sim \mathsf{N}(-1, 1)$ and $Y \sim \mathsf{N}(1, 1)$.

(c) Write the expected value of Z in terms of α, the expected value of X, and the expected value of Y.

(d) Show that if X and Y are continuous random variables, then so is Z, and find its density function.

(e) Identify the cdf of Z if $X \sim \mathsf{Bern}(2/3)$, $Y \sim \mathsf{Unif}([-1, 3])$, and $\alpha = 1/2$, and plot its graph. Is Z discrete or continuous?

(optional) (f) Let $F : \mathbb{R} \to [0, 1]$ be a cumulative distribution function that is differentiable at all but at most countably many points. Prove that F is a mixture of a continuous and a discrete distribution.

Problem 9 (Simulating distributions using a uniform RV)**.** In this exercise, you will explore how a random variable can be simulated using a uniform random variable.

Let U be a continuous random variable uniform distributed over the interval $[0, 1]$.

(a) Let $p \in [0, 1]$. Define a new random variable X, where $X = 1$ if $U \leq p$ and $X = 0$ if $U > p$. What is the distribution of X?

(b) Define a random variable X as a function of U such that $\mathbb{P}(X = -1) = 1/6$, $\mathbb{P}(X = 0) = 1/2$, and $\mathbb{P}(X = 1) = 1/3$.

(c) Generalize the idea of the previous part to prove that every discrete probability distribution can be realized as a function of a uniform random variable. More specifically, prove that for every discrete random variable X, there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(U)$ has the same distribution as X.

(d) Let $F : \mathbb{R} \to (0, 1)$ be a continuous and strictly increasing cumulative distribution function. Prove that the random variable $X := F^{-1}(U)$ has distribution F.
[*Note*: You may want to start by verifying that F is indeed bijective (hence invertible).]

(optional) (e) Generalize the idea of the previous part to prove that every probability distribution (discrete, continuous, or else) can be realized as a function of a uniform random variable. More specifically, show that for every cumulative distribution function $F : \mathbb{R} \to [0, 1]$, the random variable $X := \widetilde{F}(U)$ has distribution F, where $\widetilde{F}(p) := \inf\{x : p \leq F(x)\}$ is the *generalized inverse* of F.