American University of Beirut

MATH/STAT 233: Probability

2022–2023 Spring

Assignment 5 (due: Friday, March 31)

Suggested (ungraded) warm-up exercises:

- Problems 7.1, 7.14, 7.23, 8.1, 8.5, 8.18 of Tijms's book

Problem 1 (Ten random numbers)**.** Problem 7.9 of Tijms's book.
[*Note*: You may find this formula `https://mathoverflow.net/a/8851/23297` helpful.]

Problem 2 (Train station appointment)**.** Problem 7.12 of Tijms's book. First set up a probability model by identifying the sample space and the probability of each event.

Problem 3 (Rolling dice I)**.** Problem 7.33 of Tijms's book.

Problem 4 (Rolling dice II)**.** Problem 8.26 of Tijms's book.

Problem 5 (Infinite monkey)**.** A monkey is sitting behind a computer typing characters one after another completely at random. For simplicity, we can assume that the keyboard contains only 42 keys: the 26 letters of the English alphabet (all in upper-case form), the 10 digits `0`, `1`, …, `9`, the 4 most basic punctuation marks (full stop, question mark, exclamation mark and colon), as well as a space key and a new-line key.

(a) Build a model for this process, assuming that the monkey will continue typing forever. Specify the sample space and identify the probability of the "simple" events.
[*Hint*: Follow the example of the Bernoulli process discussed in class.]

(b) Prove that, *almost surely* (i.e., with probability 1), if we wait long enough, the monkey will eventually type the sentence:

THE NIMBLE FOX LICKS THE AVOCADOS IN THE FRIDGE.

[*Hint*: It may be helpful to divide the sequence of characters typed by the monkey into blocks of length 48.]

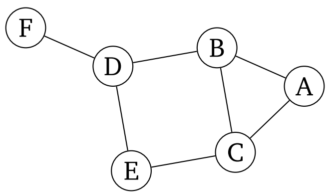
(c) Prove that, in fact, the above sentence will almost surely appear infinitely many times.

(d) State a theorem that generalizes the latter result.

Problem 6 (Particle on a ring)**.** (This problem is in the context of Problem 6 of the first midterm.)
A particle is moving at random on a ring with n points, jumping at each step, either one step counter-clockwise or one step clockwise, with probabilities p and $1-p$ respectively, where $0 < p < 1$. Prove that, *almost surely* (i.e., with probability 1), the particle will eventually make a full turn (in fact, infinitely many full turns) around the ring. [*Hint*: Express the movement of the particle in terms of an infinite coin-flipping experiment (i.e., a Bernoulli process) and apply the result of Problem 5.]

Problem 7 (Unfortunate pedestrian)**.** Let $E_1 \subseteq E_2 \subseteq \cdots$ be an infinite chain of events in a probability model $(\Omega, \mathbb{P})$, and suppose there exists an $\varepsilon > 0$ such that $\mathbb{P}(E_{n+1} \mid E_n^{\mathsf{c}}) \geq \varepsilon$ for every $n \in \{1, 2, \ldots\}$. Prove that *almost surely* (i.e., with probability 1), at least one of the events $E_1, E_2, \ldots$ will occur.

Problem 8 (Random walk on a connected graph)**.** A drunkard wanders around a town. At each junction, he takes one of the streets leading to the junction at random (possibly the one he just came from) and walks along it until he reaches the next junction. The following is the schematic map of the streets and the junctions in the town.



Prove that, no matter where the drunkard starts, *almost surely* (i.e., with probability 1), he will eventually visit all the six junctions. [*Hint*: Use the unfortunate pedestrian principle (Problem 7). Ask for a further hint if you need it.]