Full Name: ______________________

Student No: ______________________

Instructor: Siamak Taati

Grade:

Read before you start:

- Please make sure you write your full name and student number.
- The exam consists of 7 questions with multiple parts and a total score of 120 points.
- During the exam, you are allowed to use your phone **only**
  - To use the app for probability distributions,
  - To use a simple calculator app,
  - To check the time.

  but **the internet access on your phone must be switched off**. Using your phone for any purpose other than the above is **not allowed**.
- The duration of the exam is 2 hours.
- **All answers require justifications**.

You can use the remainder of this page as scratch paper.

1. (10 points) [Bernoulli RVs and their XOR]
Let X and Y be two independent Bernoulli random variables with parameters p and q, respectively, where $p, q \in (0, 1)$. Consider a new random variable Z, where

$$Z := \begin{cases} 1 & \text{if } X \neq Y, \\ 0 & \text{if } X = Y. \end{cases}$$

(a) Find all pairs of values for p and q for which $\mathbb{P}(Z = 1) = 1/2$.

(b) Find all pairs of values for p and q for which X and Z are independent.

2. (10 points) [Joint distribution]
Let X and Y be random variables with joint probability density function (joint pdf)

$$f(x,y) = \begin{cases} \dfrac{1}{x} & \text{if } 0 < y < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal densities of X and Y.

(b) What is the expected area of a rectangle with side lengths X and Y?

3. (30 points) [Students and cards]
A group of 100 students is divided into two groups, A and B, with 55 students in A and 45 students in B. Each student in A is asked to pick a number from $\{0, 1, 2\}$ at random and write it on a card. Similarly, each student in B is asked to pick a number from $\{0, -1, -2\}$ at random and write it on a card. The cards are then collected and shuffled.

We pick one of the cards at random.

(a) What is the probability that the number on the card is 0?

(b) If the number on the card turns out to be 0, what is the probability that it belongs to a student from group A?

Let $\overline{X}$ be the average of all the 100 collected numbers.

(c) Compute the expected value and variance of $\overline{X}$

(d) What can you say about the distribution of $\overline{X}$?
[*Hint*: Denote the averages over A and B by $\overline{X}_A$ and $\overline{X}_B$, respectively, and write $\overline{X}$ in terms of $\overline{X}_A$ and $\overline{X}_B$. What can you say about the distribution of $\overline{X}_A$ and $\overline{X}_B$?]

(e) Compute (or approximate) the probability that $\overline{X} > 0$.

(f) Would you be surprised if $\overline{X}$ turns out to be -0.02? What if it turns out to be -0.2? Quantify your answers.

4. (15 points) [Job interviews]
A company has two job openings, one for a *software developer* and one for a *programmer*. You are one of the 8 applicants called for an interview. The *developer* position requires a score of 5 on the list of desired criteria, while the *programmer* position requires a score of 4 or higher. You and two other applicants have scored 5 and the rest have scored 4. The candidates for the *developer* position (i.e., those with score 5) will also have to take a test. We assume that the result of each test is random with a 50% chance of success, and that the test results of different candidates are independent. The applicants are interviewed one by one, in random order, according to the following protocol, until the two positions are filled:

- The first 5-scoring interviewee who passes the test is offered the *developer* position. (If none of the three 5-scoring candidates passes the test, the position is left open.)
- The first interviewee who is not offered the *developer* position is automatically offered the *programmer* position.

For instance, if the scores and test results are the following:

scores:	4	5	4	4	5	$\cdots$
test results:		$\times$			$\checkmark$	$\cdots$

then the programmer job is offered to the first interviewee and the developer job to the 5th. We assume that any candidate who receives an offer accepts it.

(a) What is the probability that you are offered the *developer* position?

(b) What is the probability that you are offered the *programmer* position?

(c) If you are 3rd on the line, what is the chance that you are offered the *developer* position? (Note that, in this case, you can be certain that you will not be offered the *programmer* position.)

5. (20 points) [Function of a RV]
Let A be the area of a circle whose radius R is uniformly distributed over the interval $(0, 1)$.

(a) What is the expected value of A?

(b) What are the possible values of A?

(c) Find the cdf of A.

(d) Find the pdf of A.

6. (15 points) [Poisson process]
During the night shift (from 8pm to 8am), patients arrive at the emergency room of a small hospital according to a Poisson process with an average rate of 2 per hour.

(a) What is the probability that the first patient arrives before 8:30pm?

(b) What is the probability that there is a gap of at least 30 minutes between the arrival times of the first and the second patients?

(c) If we know that exactly one patient visits during the first hour, what is the probability that that patient visits before 8:30pm?

7. (20 points) [Estimating a signal]
In a communication channel, messages are transmitted from a point A to a point B as electric signals. Due to noise, a signal with value μ transmitted from A is received as $\mu + Z$ at B, where Z is a normal random variable with mean 0 and variance 4. (The noise added to each transmission is independent of the previous ones.) To reduce error, the sender transmits the same value n times. Let $X_1, X_2, \dots, X_n$ be the received values. We wish to estimate the transmitted value μ based on $X_1, X_2, \dots, X_n$.
Let $\overline{X}_n \coloneqq \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$ be the average of the received values.

   (a) What is the distribution of $\overline{X}_n$?

   (b) Assuming $n = 9$, find a margin of error r such that the interval $\overline{X}_9 \pm r$ captures the true signal value μ with 99% probability.

   (c) Assuming $n = 9$ and the received values are

$$6.3, \quad 4.5, \quad 5.6, \quad 5.8, \quad 2.2, \quad 7.8, \quad 2.5, \quad 5.8, \quad 4.0$$

   form a 99% confidence interval for μ.

   (d) Choose n so as to reduce the margin of error to 0.1 while keeping the confidence level at least 99%.

You can use this sheet as extra space for your solutions.

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