Full Name: ______________________

Student No: ______________________

Grade:

For each answer, **provide a short argument**.

1. (3 points) Suppose X and Y are two random variables with joint probability density function

$$f(x,y) := \begin{cases} 4\mathrm{e}^{-2y} & \text{if } 0 < x < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal distribution of X.

Answer: $\mathsf{Exp}(\lambda = 2)$

Solution: Let us calculate f_X, the pdf of X. For $x < 0$, clearly $f_X(x) = 0$. For $x > 0$,

$$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)\,\mathrm{d}y = \int_x^{\infty} 4\mathrm{e}^{-2y}\,\mathrm{d}y = -2\mathrm{e}^{-2y}\big|_x^{\infty} = 2\mathrm{e}^{-2x}\ .$$

Hence, X has an exponential distribution with a rate of 2.

(b) Find the marginal distribution of Y.

Answer: $\mathsf{Gamma}(\alpha = 2, \lambda = 2)$

Solution: We calculate f_Y, the pdf of Y. Again, for $y < 0$, we have $f_Y(y) = 0$. For $y > 0$,

$$f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)\,\mathrm{d}x = \int_0^{y} 4\mathrm{e}^{-2y}\,\mathrm{d}x = 4\mathrm{e}^{-2y}x\big|_0^{y} = 4y\mathrm{e}^{-2y}\ .$$

You may recognize (though it is fine if you do not) that this is the pdf of a gamma distribution with shape 2 and rate 2.

(c) Find the probability that $Y > 2X$.

Answer: $1/2$

Solution: We have

$$\begin{aligned} \mathbb{P}(Y > 2X) &= \iint\limits_{(x,y):y>2x} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{x=-\infty}^{\infty}\int_{y=2x}^{\infty} f(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{x=0}^{\infty}\int_{y=2x}^{\infty} 4\mathrm{e}^{-2y}\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{\infty}\Big(-2\mathrm{e}^{-2y}\big|_{2x}^{\infty}\Big)\,\mathrm{d}x \\ &= \int_{x=0}^{\infty} 2\mathrm{e}^{-4x}\,\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-4x}\big|_0^{\infty} = \frac{1}{2}\ . \end{aligned}$$

We could have integrated first with respect to x and then with respect to y, but that would have made the computation more elaborate, requiring integration by parts.

2. (3 points) The weight of a potato can be thought of as a random variable with a mean of 170 grams and a standard deviation of 15 grams. A sack is filled with 50 potatoes.

(a) What are the mean and standard deviations of the weight of the potato sack?

Answer: mean $= 8.5\,\mathrm{Kg}$
st. dev. $\approx 0.10607\,\mathrm{Kg}$

Solution: Let $X_1, X_2, \dots, X_{50}$ be the weights of the 50 potatoes in the sack. These are independent and identically distributed random variables with mean $170\,\mathrm{g}$ and standard deviation $15\,\mathrm{g}$. The weight of the full sack is $S = X_1 + X_2 + \cdots X_{50}$. (We ignore the weight of the sack itself.) We have

$$
\begin{aligned}
\mathbb{E}[S] &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_{50}] && \text{(linearity of expectation)} \\
&= 50 \times 170 = 8500\,\mathrm{g}\ . \\
\mathbb{V}\mathrm{ar}[S] &= \mathbb{V}\mathrm{ar}[X_1] + \mathbb{V}\mathrm{ar}[X_2] + \cdots + \mathbb{V}\mathrm{ar}[X_{50}] && \text{(independence)} \\
&= 50 \times 15^2\ . \\
\mathbb{SD}[S] &= \sqrt{\mathbb{V}\mathrm{ar}[S]} = \sqrt{50} \times 15 \\
&\approx 106.07\,\mathrm{g}\ .
\end{aligned}
$$

(b) What is the (approximate) probability that the sack weighs less than 8.4 kilograms?

Answer: ≈ 0.1729

Solution: By the central limit theorem, S is approximately normal with mean and standard deviation derived in the previous part. Using the app for $\mathsf{N}(\mu = 8.5, \sigma^2 = 0.10607^2)$, we find $\mathbb{P}(S < 8.4\,\mathrm{Kg}) \approx 0.1729$.

(c) What is the (approximate) probability that the average weight of 4 similarly packed sacks is less than 8.4?

Answer: ≈ 0.02968

Solution: Let S_1, S_2, S_3, S_4 be the weights of the 4 potato sacks. These are independent and identically distributed random variables. Let $R \coloneqq (S_1 + S_2 + S_3 + S_4)/4$ be their average.

Following the previous part, each of S_1, S_2, S_3, S_4 is approximately normally distributed with mean $8.5\,\mathrm{Kg}$ and standard deviation $0.10607\,\mathrm{Kg}$. The average of independent normal random variables is again normal, hence R is also approximately normal. Its parameters are

$$\begin{aligned}
\mathbb{E}[R] &= \frac{1}{4}\big(\mathbb{E}[S_1] + \mathbb{E}[S_2] + \mathbb{E}[S_3] + \mathbb{E}[S_4]\big) && \text{(linearity of expectation)} \\
&= 8.5\,\mathrm{Kg}\ . \\
\mathbb{V}\mathrm{ar}[R] &= \frac{1}{16}\big(\mathbb{V}\mathrm{ar}[S_1] + \mathbb{V}\mathrm{ar}[S_2] + \mathbb{V}\mathrm{ar}[S_3] + \mathbb{V}\mathrm{ar}[S_4]\big) && \text{(independence)} \\
&\approx \frac{1}{4} \times 0.10607^2\ . \\
\mathbb{SD}[R] &= \sqrt{\mathbb{V}\mathrm{ar}[R]} \approx \frac{1}{2} \times 0.10607 \\
&= 0.053035\,\mathrm{Kg}\ .
\end{aligned}$$

Using the app for the normal distribution, we get $\mathbb{P}(R < 8.4) \approx 0.02968$.

3. (0 points) On the scale of 0–6, what is your estimate of your grade on this quiz?