Full Name: ____________________

Student No: ____________________

Grade:

For each answer, **provide a short argument**.

1. (1 point) Let U and V be independent standard normal random variables. What is the distribution of $3U - 4V + 5$?

Solution: Recall that the normal distribution is stable, that is:

- The scaled and shifted version of any normal RV is again nomal,
- The sum of any two independent normal RVs is again normal.

Hence, $X := 3U - 4V + 5$ is a normal random variable. We only need to find its two parameters, namely its mean and its variance. The mean of X is

$$\begin{aligned}
\mu_X = \mathbb{E}[X] &= \mathbb{E}[3U - 4V + 5] \\
&= 3\,\mathbb{E}[U] - 4\,\mathbb{E}[V] + 5 && \text{(linearity of expectation)} \\
&= 3 \times 0 - 4 \times 0 + 5 && \text{(U and V are standard normal)} \\
&= 5\ .
\end{aligned}$$

The variance of X is

$$\begin{aligned}
\sigma_X^2 = \mathbb{V}\mathrm{ar}[X] &= \mathbb{V}\mathrm{ar}[3U - 4V + 5] \\
&= \mathbb{V}\mathrm{ar}[3U - 4V] && \text{(shifting does not affect the variance)} \\
&= \mathbb{V}\mathrm{ar}[3U] + \mathbb{V}\mathrm{ar}[-4V] && \text{($3U$ and $-4V$ are independent)} \\
&= 9\,\mathbb{V}\mathrm{ar}[U] + 16\,\mathbb{V}\mathrm{ar}[V] && \text{(variance of the scaled version of a RV)} \\
&= 9 \times 1 + 16 \times 1 && \text{(U and V are standard normal)} \\
&= 25\ .
\end{aligned}$$

In summary, X is normal with mean 5 and variance 25.

2. (1 point) In a certain city, fatal traffic accidents occur according to a Poisson process with an average rate of 7 per year. What is the probability that the 2nd fatal accident in 2024 occurs by the end of January? [*Note*: 2024 will be a leap year, which means it will have 366 days. January has 31 days.]

Answer:
≈ 0.11957

Solution:
Approach 1:
Let S denote the arrival time of the 2nd accident in 2024, measured from the beginning of the year. We are looking for $\mathbb{P}(S \leq 31\,\text{days})$. Note that S is a gamma random variable with shape $r = 2$ and

rate $\lambda = 7$ per year because the accidents occur according to a Poisson process. Thus,

$$\begin{aligned}
\mathbb{P}(S \le 31\,\text{days}) &= \int_{-\infty}^{31/366} f_S(x)\,\mathrm{d}x \\
&= \int_0^{31/366} 7^2 x \mathrm{e}^{-7x}\,\mathrm{d}x && \text{(using the pdf of Gamma}(r=2,\lambda=7)) \\
&= -7x\mathrm{e}^{-7x}\Big|_0^{\frac{31}{366}} + 7\int_0^{\frac{31}{366}} \mathrm{e}^{-7x}\,\mathrm{d}x && \text{(integration by parts)} \\
&\approx 0.11957\,.
\end{aligned}$$

Approach 2:
A simpler approach that does not require integration is to note that

- The event that the 2nd accident in 2024 occurs by the end of January can equivalently be described as the event that during January, there are at least 2 accidents.

More precisely, let $N(\text{January})$ denote the number of accidents during January 2024. Then, the two events $\{S \le 31\,\text{days}\}$ and $\{N(\text{January}) \ge 2\}$ coincide. Now, note that $N(\text{January})$ is a Poisson random variable with mean

$$\mu = (7\,\text{per year}) \times (31\,\text{days}) = 7 \times \frac{31}{366} = \frac{217}{366}\,.$$

Thus,

$$\begin{aligned}
\mathbb{P}(S \le 31\,\text{days}) &= \mathbb{P}\big(N(\text{January}) \ge 2\big) \\
&= 1 - \mathbb{P}\big(N(\text{January}) = 0\big) - \mathbb{P}\big(N(\text{January}) = 1\big) \\
&= 1 - \mathrm{e}^{-\mu} - \mathrm{e}^{-\mu}\frac{\mu^1}{1!} \\
&= 1 - \mathrm{e}^{-217/366}(1 + \frac{217}{366}) \\
&\approx 0.11957\,.
\end{aligned}$$

3. (3 points) Let T be an exponential random variable with rate 5.

(a) What is the probability that $-1 \le T \le 1$?

Answer:
≈ 0.99326

Solution: We have

$$\begin{aligned}
\mathbb{P}(-1 \le T \le 1) &= \int_{-1}^{1} f_T(t)\,\mathrm{d}t \\
&= \int_0^1 5\mathrm{e}^{-5t}\,\mathrm{d}t && \text{(using the pdf of Exp}(\lambda=5)) \\
&= 1 - \mathrm{e}^{-5} \\
&\approx 0.99326
\end{aligned}$$

Now, consider the random variable $X = \mathrm{e}^{-T}$.

(b) What are the possible values of X?

> **Solution:** The interval $(0, 1]$.
> Namely, since the possible values of T are all the real numbers between 0 and $+\infty$, the possible values of $X = \mathrm{e}^{-T}$ are all the numbers between 0 and 1 (including 1 but excluding 0).

(c) Find the probability density function of X.

> **Solution:** As usual, it is easier to start with the cumulative density function. For $x \in (0, 1]$ (the possible values), we have
>
> $$\begin{aligned} F_X(x) &= \mathbb{P}(X \le x) \\ &= \mathbb{P}(\mathrm{e}^{-T} \le x) \\ &= \mathbb{P}\big(-T \le \ln(x)\big) && \text{(the events } \{-T \le \ln(x)\} \text{ and } \{\mathrm{e}^{-T} \le x\} \text{ are the same)} \\ &= \mathbb{P}\big(T \ge -\ln(x)\big) && \text{(the events } \{T \le -\ln(x)\} \text{ and } \{-T \le \ln(x)\} \text{ are the same)} \\ &= 1 - \mathbb{P}\big(T < -\ln(x)\big) \\ &= 1 - F_T\big(-\ln(x)\big) && \text{(the cdf of } T \text{ is continuous)} \\ &= 1 - \Big(1 - \mathrm{e}^{5\ln(x)}\Big) && \text{(using the cdf of } \mathsf{Exp}(\lambda = 5)\text{)} \\ &= x^5 \,. \end{aligned}$$
>
> Differentiating with respect to x, we get $f_X(x) = 5x^4$ for $x \in (0, 1)$. Thus, the pdf of X is
>
> $$f_X(x) = \begin{cases} 5x^4 & \text{if } 0 < x < 1, \\ 0 & \text{if } x < 0 \text{ or } x > 1. \end{cases}$$

4. (0 points) On the scale of 0–5, what is your estimate of your grade on this quiz?