Full Name: ______________________

Student No: ______________________

Instructor: Siamak Taati

Grade:

Read before you start:

- Please make sure you write your full name and student number.
- The exam consists of 9 questions, most with multiple parts, and a total score of 155 points. All answers require justifications.
- During the exam, you are allowed to use your phone to access the app for probability distributions (and possibly a calculator app), but the internet access of your phone must be switched off.
- The duration of the exam is 2 hours.

You can use the remainder of this page as scratch paper.

1. (15 points) Determine which of the following statements is True and which is False. In each case, give a short justification.

_____ If A and B are two independent events, then necessarily $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

_____ If A and B are two events with non-zero probabilities, then $\mathbb{P}(A \cap B \mid A \cup B) \leq \mathbb{P}(A \mid B)$.

_____ If X and Y are discrete random variables with probability mass functions $p_X(a)$ and $p_Y(a)$ respectively, then $X + Y$ is also discrete and has probability mass function $p_X(a) + p_Y(a)$.

2. (15 points) A jar contains 10 blue balls and 5 red balls. We draw the balls one after another at random without replacement until there is no ball left in the jar.

   (a) What is the probability that the last ball is red?

   (b) What is the probability that no two consecutive balls we draw are red?

   (c) If the 2nd ball is red, what is the probability that the 1st ball has also been red?

3. (25 points) Let U be a random variable uniformly distributed over the interval $[0, 1]$, and let $X = \frac{1}{1+U}$.

(a) Find the expected value of X.

(b) What are the possible values of X?

(c) Find the cdf of X.

(d) Find the pdf of X.

(e) What is the probability that $1/4 < X < 3/4$?

4. (15 points) Let $X \sim \mathsf{N}(\mu = 2, \sigma^2 = 1)$ and $Y \sim \mathsf{N}(\mu = -1, \sigma^2 = 2^2)$ be two independent normal random variables. What is the distribution of $-2X + Y + 1$?

5. (15 points) Two random variables X and Y have joint probability density

$$f(x,y) = \begin{cases} x+y & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal distributions of X and Y.

(b) Are X and Y independent?

(c) Compute $\mathbb{P}(X+Y>1)$.

6. (20 points) In a digital communication channel, messages are sent and received as strings of 0s and 1s. Due to noise, the transmission of each bit of information is subject to error (i.e., receiving a 0 instead of a 1, or vice versa). We assume that every bit is transmitted incorrectly with probability 0.01, independently of the other bits.

   (a) What is the probability that a message with 100 bits is transmitted without error?

   (b) What is the expected number of errors in a message with 10000 bits?

   (c) What is the variance of the number of errors in a message with 10000 bits?

   (d) Find the (approximate) probability that the number of errors in a message with 10000 bits is less than 85?

7. (25 points) A random variable X has a uniform distribution over the interval $[0, \beta]$, where β is an unknown parameter. Consider the following point estimators:

$$\widehat{\beta} \coloneqq 2\overline{X}_n \qquad\qquad \widetilde{\beta} \coloneqq \max\{X_1, X_2, \ldots, X_n\}$$

for β based on an independent sample $X_1, X_2, \ldots, X_n$. (Here, $\overline{X}_n \coloneqq (X_1 + X_2 + \cdots + X_n)/n$ stands for the sample mean.)

(a) Argue that $\widehat{\beta}$ is consistent.

(b) Argue that $\widetilde{\beta}$ is consistent.

(c) Is $\widehat{\beta}$ biased?

Let $n = 5$, and suppose we observe the values

$$X_1 = 0.2312 \,, \qquad X_2 = 3.8882 \,, \qquad X_3 = 2.0016 \,, \qquad X_4 = 2.1474 \,, \qquad X_5 = 0.1431 \,.$$

(d) Compute the estimates provided by $\widehat{\beta}$ and $\widetilde{\beta}$.

(e) Given the above observed values, is the estimate provided by $\widehat{\beta}$ reasonable?

8. (5 points) At AUBMC, on average, 3 babies are born per day. The arrival times of the babies can be modeled by a Poisson process. What is the probability that, on Sunday, at least 2 babies arrive before noon? (We take the midnight as the start of the day.)

9. (20 points) An ear thermometer measures the body temperature (more specifically, the temperature inside your ear canal) using an infrared sensor. Like any other measurement device, an ear thermometer is subject to errors. The reading on the thermometer can be modeled as

$$T = \theta + R\,,$$

where θ is the actual (non-random, but unknown) body temperature and R is the random error. We assume that R is normally distributed with mean 0 and an unknown standard deviation σ.

(a) Explain why repeating the measurement a few times and taking the average of the readings gives a better estimate for θ than just a single measurement.

Let $T_1, T_2, \ldots, T_5$ be the results of 5 separate (independent) measurements, and

$$\overline{T} = \frac{T_1 + T_2 + \cdots + T_5}{5} \qquad \text{and} \qquad S^2 = \frac{1}{4}\sum_{i=1}^{5}(T_i - \overline{T})^2$$

their sample mean and sample variance.

(b) Compute $\mathbb{E}[\overline{T}]$ and $\mathbb{V}\mathrm{ar}[\overline{T}]$ in terms of the parameters of the model.

(c) Based on these 5 measurements, construct an interval estimator for θ with confidence level 98%.

Suppose you measure your temperature 5 times in a row and find values

$$36.9\ °\mathrm{C} \qquad 36.8\ °\mathrm{C} \qquad 37.0\ °\mathrm{C} \qquad 36.8\ °\mathrm{C} \qquad 36.8\ °\mathrm{C}$$

which have mean 36.86 °C and standard deviation 0.08944 °C.

(d) Use the latter interval estimator to give a 98% confidence interval for your body temperature.

You can use this page as extra space for your solutions.