Full Name: ______________________

Student No: ______________________

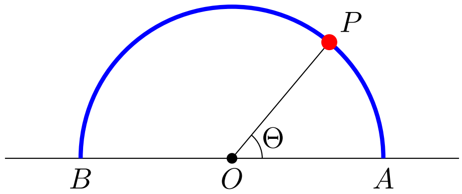
Grade:

Read before you start:

- Please make sure you write your full name and student number.
- The exam consists of 6 questions, some with multiple parts, and a total score of 70 points. All answers require justifications.
- The duration of the exam is 1h30.

You can use the remainder of this page as scratch paper.

1. (15 points) A point P is picked uniformly at random from a semicircle. Let Θ denote the angle of this point relative to one of the endpoints of the semicircle.



(a) What is the probability that $\sin(\Theta) > \frac{1}{2}$?

Solution: The angle Θ is uniformly distributed over the interval $[0, \pi]$. Note that for $0 \leq \theta \leq \pi$, the condition $\sin(\theta) > 1/2$ is equivalent $\pi/6 < \theta < 5\pi/6$. Thus,

$$\mathbb{P}(\sin(\Theta) > 1/2) = \mathbb{P}(\pi/6 < \theta < 5\pi/6) = \frac{5\pi/6 - \pi/6}{\pi - 0} = \boxed{2/3}\,.$$

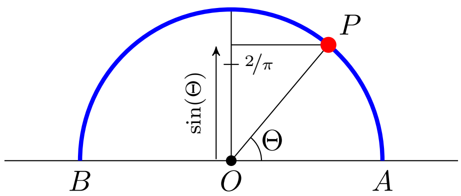
(b) What is the expected value of $\sin(\Theta)$?

Solution: The angle Θ is uniformly distributed on $[0, \pi]$, hence its pdf is

$$f_\Theta(\theta) = \begin{cases} 1/\pi & \text{if } 0 < \theta < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

We can use the distribution of Θ to compute

$$\mathbb{E}[\sin(\Theta)] = \int_{-\infty}^{\infty} \sin(\theta) f_\Theta(\theta)\, \mathrm{d}\theta = \int_0^\pi \sin(\theta) \frac{1}{\pi}\, \mathrm{d}\theta = \left. \frac{-\cos(\theta)}{\pi} \right|_{\theta=0}^{\pi} = \boxed{2/\pi} \approx 0.6366\,.$$



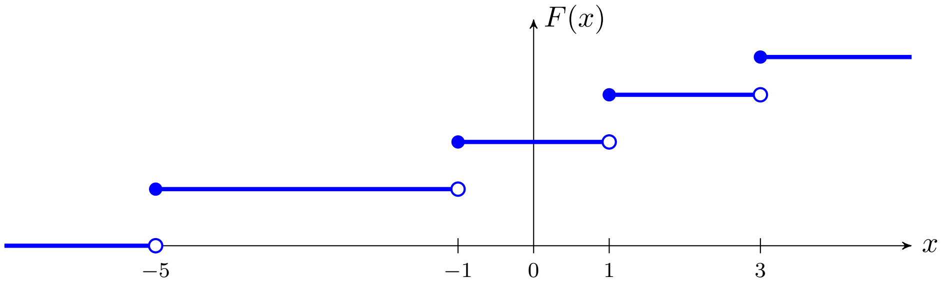
(c) What is the expected value of $\cos(\Theta)$?

Solution:
Approach 1: Direct computation similar to the previous part leads to the answer $\mathbb{E}[\cos(\Theta)] = 0$.

Approach 2: $\mathbb{E}[\cos(\Theta)] = 0$ by symmetry. Indeed, both $\cos(\theta)$ and the distribution of Θ are symmetric about $\theta = \pi/2$.

2. (20 points) A random variable X has the following cumulative distribution function:

$$F(x) = \begin{cases} 0 & \text{if } x < -5, \\ 0.3 & \text{if } -5 \leq x < -1, \\ 0.55 & \text{if } -1 \leq x < 1, \\ 0.8 & \text{if } 1 \leq x < 3, \\ 1 & \text{if } 3 \leq x. \end{cases}$$



(a) What is the probability that $X > 0$?

Solution: We have $\mathbb{P}(X > 0) = 1 - \mathbb{P}(X \leq 0) = 1 - F(0) = 1 - 0.55 = \boxed{0.45}$.

(b) What is the probability that $1 \leq X \leq 3$?

Solution: We have

$$\mathbb{P}(1 \leq X \leq 3) = \mathbb{P}(X \leq 3) - \mathbb{P}(X < 1) = \mathbb{P}(X \leq 3) - \mathbb{P}(X \leq 1) + \mathbb{P}(X = 1) .$$

We know $\mathbb{P}(X \leq 3) = F(3) = 1$ and $\mathbb{P}(X \leq 1) = F(1) = 0.8$. To find $\mathbb{P}(X = 1)$, we note that at $x = 1$, the cdf $F(x)$ has a jump from 0.55 to 0.8. Hence, $\mathbb{P}(X = 1) = 0.8 - 0.55 = 0.25$. We conclude that $\mathbb{P}(1 \leq X \leq 3) = 1 - 0.8 + 0.25 = \boxed{0.45}$.
Alternatively, we may note that X is a discrete random variable with possible values $-5, -1, 1, 3$, find its probability mass function (see the next part), and compute

$$\mathbb{P}(1 \leq X \leq 3) = \mathbb{P}(X = 1 \text{ or } X = 3) = \mathbb{P}(X = 1) + \mathbb{P}(X = 3) = 0.2 + 0.25 = \boxed{0.45} .$$

(c) Find the expected value of X.

Solution: From its cdf, we observe that X is a discrete random variable with possible values $-5, -1, 1, 3$. (Indeed, $F(x)$ has jumps at these values and is constant between every two consecutive jump.) The height of the jump gives the probability of each possible value:

$$\mathbb{P}(X = -5) = 0.3 - 0 = 0.3 , \qquad \mathbb{P}(X = -1) = 0.55 - 0.3 = 0.25 ,$$
$$\mathbb{P}(X = 1) = 0.8 - 0.55 = 0.25 , \qquad \mathbb{P}(X = 3) = 1 - 0.8 = 0.2 .$$

(As a sanity check, $0.3 + 0.25 + 0.25 + 0.2 = 1$.) The expected value can now be calculated as

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{P}(X = -5) \times (-5) + \mathbb{P}(X = -1) \times (-1) + \mathbb{P}(X = 1) \times 1 + \mathbb{P}(X = 3) \times 3 \\ &= 0.3 \times (-5) + 0.25 \times (-1) + 0.25 \times 1 + 0.2 \times 3 = \boxed{-0.9} . \end{aligned}$$

(d) Find the variance of X.

Solution: We use the equality $\mathbb{V}\mathrm{ar}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$. We already know $\mathbb{E}[X]$. Using the probability mass function of X, we can calculate

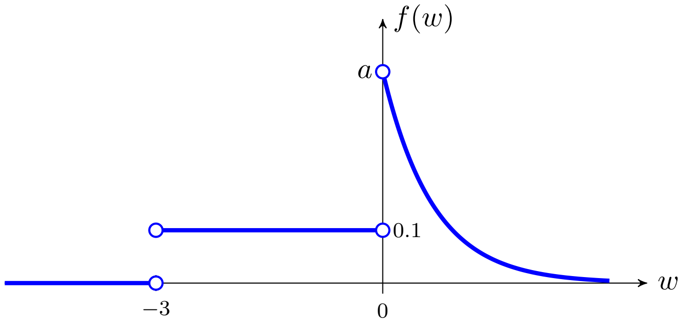
$$\begin{aligned}\mathbb{E}[X^2] &= \mathbb{P}(X=-5)\times(-5)^2 + \mathbb{P}(X=-1)\times(-1)^2 + \mathbb{P}(X=1)\times 1^2 + \mathbb{P}(X=3)\times 3^2 \\ &= 0.3\times 25 + 0.25\times 1 + 0.25\times 1 + 0.2\times 9 = 9.8\ .\end{aligned}$$

Therefore, $\mathbb{V}\mathrm{ar}[X] = 9.8 - (-0.9)^2 = \boxed{8.99}$.

3. (15 points) A continuous random variable W has the following probability density function:

$$f(w) = \begin{cases} 0 & \text{if } w < -3, \\ 0.1 & \text{if } -3 < w < 0, \\ a\,\mathrm{e}^{-2w} & \text{if } 0 < w, \end{cases}$$

where a is an unknown constant.



The sketch is not up to scale!

(a) Find a.

Solution: In order for $f(w)$ to be a probability density function, its integral from $-\infty$ to ∞ (i.e., the total area under its graph) must be 1, that is,

$$1 = \int_{-\infty}^{\infty} f(w)\,\mathrm{d}w = \int_{-3}^{0} 0.1\,\mathrm{d}w + \int_{0}^{\infty} a\mathrm{e}^{-2w}\,\mathrm{d}w = 0.3 + 0.5a$$

Solving for a, we obtain $a = \boxed{1.4}$.

(b) What is the probability that $W \le 1$?

Solution: The probability of $W \le 1$ is the area under the graph of $f(w)$ to the left of $w = 1$, that is,

$$\begin{aligned}\mathbb{P}(W \le 1) &= \int_{-\infty}^{1} f(w)\,\mathrm{d}w = \int_{-3}^{0} 0.1\,\mathrm{d}w + \int_{0}^{1} 1.4\mathrm{e}^{-2w}\,\mathrm{d}w \\ &= 0.3 + 1.4\times \left.\frac{\mathrm{e}^{-2w}}{-2}\right|_{0}^{1} = \boxed{1 - 0.7\mathrm{e}^{-2}} \approx 0.9053\ .\end{aligned}$$

Alternatively, $\mathbb{P}(W \le 1) = 1 - \mathbb{P}(W > 1) = 1 - \int_{1}^{\infty} 1.4\mathrm{e}^{-2w}\,\mathrm{d}w = \boxed{1 - 0.7\mathrm{e}^{-2}}$.

(c) What is the expected value of W?

Solution: We have

$$\mathbb{E}[W] = \int_{-\infty}^{\infty} w f(w)\,\mathrm{d}w = \underbrace{\int_{-3}^{0} 0.1w\,\mathrm{d}w}_{\clubsuit} + \underbrace{\int_{0}^{\infty} 1.4w\mathrm{e}^{-2w}\,\mathrm{d}w}_{\diamondsuit} .$$

Let us compute the two integrals separately. The first one is

$$\clubsuit = 0.1 \times \frac{1}{2}w^2\Big|_{-3}^{0} = -0.45 .$$

The second integral can be computed using integration by parts (with $u = 1.4w$ and $v = \frac{1}{-2}\mathrm{e}^{-2w}$):

$$\diamondsuit = \overbrace{\cancel{1.4w \times \frac{1}{-2}\mathrm{e}^{-2w}\Big|_{0}^{\infty}}}^{0} - \int_{0}^{\infty} 1.4 \times \frac{1}{-2}\mathrm{e}^{-2w}\,\mathrm{d}w = \frac{1.4}{-4}\mathrm{e}^{-2w}\Big|_{0}^{\infty} = \frac{1.4}{4} = 0.35 .$$

We conclude that $\mathbb{E}[W] = -0.45 + 0.35 = \boxed{-0.1}$.

4. (5 points) Let X be a normal random variable with mean 3 and variance 4. Express the probability that $2 \leq X \leq 5$ either in terms of the cumulative distribution function Φ of the standard normal distribution, or as an integral.

Solution: By definition, a normal random variable is a shifted and scaled version of a standard normal random variable. In particular, X can be written as $X = \sigma Z + \mu$, where Z is a standard normal random variable, and μ and σ are the mean and standard deviation of X. Here, $\mu = 3$ and $\sigma = \sqrt{4} = 2$. Therefore,

$$\mathbb{P}(2 \leq X \leq 5) = \mathbb{P}(2 \leq 2Z + 3 \leq 5) = \mathbb{P}(-1/2 \leq Z \leq 1) = \boxed{\Phi(1) - \Phi(-1/2)} .$$

5. (5 points) Let G be a discrete random variable with the following probability mass function:

$$p(n) = \begin{cases} (5/6) \times (2/3)^n & \text{if } n = 1, 3, 5, 7, \dots \text{ (a positive odd integer)}, \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected value of G.

Solution:

Approach 1 (explicit computation):
We have

$$\begin{aligned} \mathbb{E}[G] &= 1 \cdot (5/6)(2/3) + 3 \cdot (5/6)(2/3)^3 + 5 \cdot (5/6)(2/3)^5 + \cdots \\ &= \sum_{k=0}^{\infty} (2k+1)(5/6)(2/3)^{2k+1} \end{aligned}$$

The above series resembles the one that arises when want to compute the expected value of a geometric random variable. To compute it, we can use the following generating function:

$$g(z) = z + z^3 + z^5 + z^7 + \cdots = \frac{z}{1 - z^2} \ .$$

The latter equality holds when the geometric series defining $g(z)$ converges, that is, when $-1 < z < 1$. Differentiating with respect to z, we obtain

$$g'(z) = 1 + 3z^2 + 5z^4 + 7z^6 + \cdots = \frac{1 + z^2}{(1 - z^2)^2} \ .$$

Comparing the latter series with the series we are trying to compute, we find that

$$\mathbb{E}[G] = (5/6)(2/3) g'(2/3) = (5/6)(2/3) \times \frac{1 + (2/3)^2}{\big(1 - (2/3)^2\big)^2} = \boxed{13/5} \, .$$

Approach 2 (connection with a geometric RV):
There is a clear similarity with the pmf of a geometric distribution. Since the possible values of G are the positive odd numbers, we may guess that $G = 2T - 1$ for some geometric random variable. Indeed, the possible values of $2T - 1$ are also the positive odd numbers. If T has parameter p, then

$$\begin{aligned} \mathbb{P}(2T - 1 = 1) &= \mathbb{P}(T = 1) = p \ , \\ \mathbb{P}(2T - 1 = 3) &= \mathbb{P}(T = 2) = (1-p)p \\ \mathbb{P}(2T - 1 = 5) &= \mathbb{P}(T = 3) = (1-p)^2 p \\ \vdots &= \vdots \\ \mathbb{P}(2T - 1 = 2k+1) &= \mathbb{P}(T = k+1) = (1-p)^k p \ . \end{aligned}$$

Comparison with the pmf of G shows a perfect match if $p = 1 - (2/3)^2 = 5/9$. We conclude that

$$\mathbb{E}[G] = \mathbb{E}[2T - 1] = 2\,\mathbb{E}[T] - 1 = 2 \times \frac{1}{5/9} - 1 = \boxed{13/5} \, .$$

Remark. In another version of the exam, the constant in front of the exponential was incorrect, leading to a nonsensical result that $\mathbb{E}[G] < 1$. Obviously, since G is always at least one and has a positive probability of being larger than 1, its expected value must be larger than 1.

6. (10 points) A box contains 10 cards numbered 1 to 10. You are offered to play the following game: At the beginning of the game, you pay an entrance fee of $\$3$. You then draw 6 cards from the box, one after another, with replacement. You receive $\$1$ for each distinct number you draw. For instance, if you draw $1, 6, 4, 4, 3, 4$, you receive $\$4$ for the distinct numbers $1, 3, 4, 6$.

   (a) Calculate your expected payoff. [*Hint*: Use the linearity of expectation.]

Solution: Let X denote your (random) payoff in dollars. We have $X = -3 + N$, where N is the number of distinct numbers you draw. We have $\mathbb{E}[X] = -3 + \mathbb{E}[N]$. To compute $\mathbb{E}[N]$, we have two options: (i) derive the distribution of N and follow the definition, (ii) write N as a sum of simpler random variables and use the linearity of expectation. We follow the second approach.
For each number $k = 1, 2, \ldots, 10$, define a Bernoulli random variable

$$X_k = \begin{cases} 1 & \text{if card number } k \text{ is drawn,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $N = X_1 + X_2 + \cdots + X_{10}$. We have

$$\mathbb{E}[X_k] = \mathbb{P}(X_k = 1) = \mathbb{P}(\text{card number } k \text{ is drawn}) = 1 - ({}^{9}\!/_{10})^6 \ .$$

Therefore,

$$\mathbb{E}[N] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_{10}] = 10\big(1 - ({}^{9}\!/_{10})^6\big) \ .$$

We conclude that the expected payoff is

$$\mathbb{E}[X] = -3 + 10\big(1 - ({}^{9}\!/_{10})^6\big) = \boxed{7 - 10 \cdot ({}^{9}\!/_{10})^6} = 1.68559 \ .$$

   (b) Is this a fair game? If not, how would you change the entrance fee in order to make it fair?

Solution: Since the expected payoff is not zero, the game is unfair. (If you repeat the game many many times, on average, you gain about $\$1.68559$ per game.)
To make the game fair, the fee must be the same as $\mathbb{E}[N] = 10\big(1 - ({}^{9}\!/_{10})^6\big) = \boxed{4.68559}$ dollars so that $\mathbb{E}[X] = 0$.