

American University of Beirut  
STAT 210: Elementary Statistics for Sciences  
2022–2023 Fall

Siamak Taati

Chapter 5  
Discrete random variables  
Part 1

# Discrete random variables

# Language of probabilities: random variables

We saw how probabilities and probability models provide a simple, precise and unambiguous language to help us discuss and reason about chance and randomness.

**Random variables** are another element of the language of probabilities. They provide the vocabulary for talking and reasoning about random quantities in a random experiment.

# Language of probabilities: random variables

## Examples of random quantities

- ▶ The height of a randomly picked student from this class
- ▶ The weight of a randomly picked bird from a species of birds
- ▶ The number of students in this class who will get an A
- ▶ The number of spam emails you will receive during November
- ▶ The time until you receive the next spam email
- ▶ Tomorrow's temperature in Beirut
- ▶ ...

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 1

Experiment: flip a coin twice.

A random variable:  $N := \#$  of heads

The value of  $N$  is determined by the outcome:

| outcome | HH | HT | TH | TT |
|---------|----|----|----|----|
| $N$     | 2  | 1  | 1  | 0  |

Possible values: 0, 1, 2.

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 2

Experiment: Roll two 6-sided dice.

A random variable:  $X :=$  sum of the numbers on the dice

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 2

Experiment: Roll two 6-sided dice.

A random variable:  $X :=$  sum of the numbers on the dice

① What are the possible values of  $X$ ?

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 2

Experiment: Roll two 6-sided dice.

A random variable:  $X :=$  sum of the numbers on the dice

Ⓚ What are the possible values of  $X$ ?

Ⓜ 2, 3, 4, ..., 12.

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 2

Experiment: Roll two 6-sided dice.

A random variable:  $X :=$  sum of the numbers on the dice

$X$ :

|                                                                                   |   |   |   |    |    |    |
|-----------------------------------------------------------------------------------|---|---|---|----|----|----|
|  | 7 | 8 | 9 | 10 | 11 | 12 |
|  | 6 | 7 | 8 | 9  | 10 | 11 |
|  | 5 | 6 | 7 | 8  | 9  | 10 |
|  | 4 | 5 | 6 | 7  | 8  | 9  |
|  | 3 | 4 | 5 | 6  | 7  | 8  |
|  | 2 | 3 | 4 | 5  | 6  | 7  |
|  |   |   |   |    |    |    |

die #2  
die #1

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 3

A random variable:

$N := \#$  of spam emails you will receive during November

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 3

A random variable:

$N := \#$  of spam emails you will receive during November

Experiment: Monitor your mailbox during November.

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 3

A random variable:

$N := \#$  of spam emails you will receive during November

Experiment: Monitor your mailbox during November.

Sample space: All possible lists of emails you could receive during November

| Spam | Time        | Sender          | Subject                         |
|------|-------------|-----------------|---------------------------------|
| *    | November 1  | Mom             | Come home early!                |
|      | November 1  | Instructor      | Reminder: midterm 2 will be ... |
|      | November 2  | Mr. Joe         | BEST deal!!!                    |
|      | November 3  | Ali             | What's up?                      |
|      | November 4  | L.              | Miss you <3                     |
|      | :           | :               |                                 |
|      | :           | :               |                                 |
|      | :           | :               |                                 |
| *    | November 30 | Mr. H. Benjamin | DEAR, BENEFICIARY!              |

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 3

A random variable:

$N := \#$  of spam emails you will receive during November

Experiment: Monitor your mailbox during November.

Sample space: All possible lists of emails you could receive during November

Q What are the possible values of  $N$ ?

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 3

A random variable:

$N := \#$  of spam emails you will receive during November

Experiment: Monitor your mailbox during November.

Sample space: All possible lists of emails you could receive during November

Q What are the possible values of  $N$ ?

A  $0, 1, 2, 3, \dots$

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 4

A random variable:

$T$  := time until you receive the next spam email  
(measured in seconds)

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 4

A random variable:

$T$  := time until you receive the next spam email  
(measured in seconds)

Experiment: Start a timer and monitor your mailbox until a spam arrives.

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 4

A random variable:

$T$  := time until you receive the next spam email  
(measured in seconds)

Experiment: Start a timer and monitor your mailbox until a spam arrives.

Sample space: ???

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 4

A random variable:

$T$  := time until you receive the next spam email  
(measured in seconds)

Experiment: Start a timer and monitor your mailbox until a spam arrives.

Sample space: ???

① What are the possible values of  $T$ ?

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 4

A random variable:

$T$  := time until you receive the next spam email  
(measured in seconds)

Experiment: Start a timer and monitor your mailbox until a spam arrives.

Sample space: ???

Q What are the possible values of  $T$ ?

A All (real) numbers larger than or equal to 0.

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 5

A random variable:

$T$  := tomorrow's temperature in Beirut  
(measured in centigrade)

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 5

A random variable:

$T$  := tomorrow's temperature in Beirut  
(measured in centigrade)

(More precisely: average of 10 different measurements at different locations during the day.)

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 5

A random variable:

$T$  := tomorrow's temperature in Beirut  
(measured in centigrade)

Experiment: ???

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 5

A random variable:

$T$  := tomorrow's temperature in Beirut  
(measured in centigrade)

Experiment: ???

Sample space: ???

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 5

A random variable:

$T :=$  tomorrow's temperature in Beirut  
(measured in centigrade)

Experiment: ???

Sample space: ???

Q What are the possible values of  $T$ ?

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Example 5

A random variable:

$T :=$  tomorrow's temperature in Beirut  
(measured in centigrade)

Experiment: ???

Sample space: ???

Q What are the possible values of  $T$ ?

A All (real) numbers between say  $-50$  and  $50$ .

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Two types of random variables

- ▶ A **discrete** random variable is one which has only finitely many or countably many possible values.
  
- ▶ A **continuous** random variable is one whose values range over an entire interval (or intervals).

# Random variables

A **random variable** within a random experiment is a variable  $X$  (often numerical) whose value is determined by the outcome of the experiment.

## Two types of random variables

- ▶ A **discrete** random variable is one which has only finitely many or countably many possible values.

For example:

- Number of heads in two flips a coin
- Sum of the numbers on two dice
- Number of spam emails during November

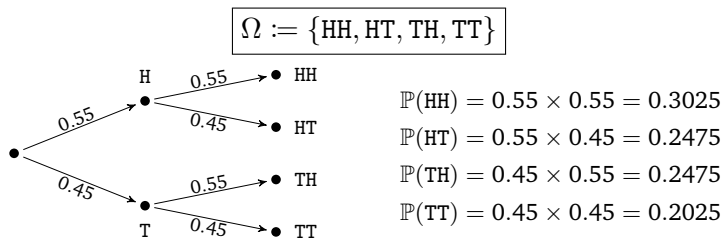
- ▶ A **continuous** random variable is one whose values range over an entire interval (or intervals).

For example:

- Time until the next spam email
- Tomorrow's temperature in Beirut

# Discrete random variables

## Example (Flipping a biased coin twice)



# Discrete random variables

## Example (Flipping a biased coin twice)

$$\Omega := \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$$

$$\mathbb{P}(\text{HH}) = 0.3025 \quad \mathbb{P}(\text{HT}) = 0.2475 \quad \mathbb{P}(\text{TH}) = 0.2475 \quad \mathbb{P}(\text{TT}) = 0.2025$$

# Discrete random variables

## Example (Flipping a biased coin twice)

$$\Omega := \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$$

$$\mathbb{P}(\text{HH}) = 0.3025 \quad \mathbb{P}(\text{HT}) = 0.2475 \quad \mathbb{P}(\text{TH}) = 0.2475 \quad \mathbb{P}(\text{TT}) = 0.2025$$

Consider the random variable:

$$N := \# \text{ of heads}$$

# Discrete random variables

## Example (Flipping a biased coin twice)

$$\Omega := \{HH, HT, TH, TT\}$$

$$\mathbb{P}(HH) = 0.3025 \quad \mathbb{P}(HT) = 0.2475 \quad \mathbb{P}(TH) = 0.2475 \quad \mathbb{P}(TT) = 0.2025$$

Consider the random variable:

$$N := \# \text{ of heads}$$

Q What is the probability that  $N$  takes each of its possible values?

# Discrete random variables

## Example (Flipping a biased coin twice)

$$\Omega := \{HH, HT, TH, TT\}$$

$$\mathbb{P}(HH) = 0.3025 \quad \mathbb{P}(HT) = 0.2475 \quad \mathbb{P}(TH) = 0.2475 \quad \mathbb{P}(TT) = 0.2025$$

Consider the random variable:

$$N := \# \text{ of heads}$$

| outcome | HH | HT | TH | TT |
|---------|----|----|----|----|
| $N$     | 2  | 1  | 1  | 0  |

Q What is the probability that  $N$  takes each of its possible values?

# Discrete random variables

## Example (Flipping a biased coin twice)

$$\Omega := \{HH, HT, TH, TT\}$$

$$\mathbb{P}(HH) = 0.3025 \quad \mathbb{P}(HT) = 0.2475 \quad \mathbb{P}(TH) = 0.2475 \quad \mathbb{P}(TT) = 0.2025$$

Consider the random variable:

$N := \# \text{ of heads}$

| outcome | HH | HT | TH | TT |
|---------|----|----|----|----|
| $N$     | 2  | 1  | 1  | 0  |

Q What is the probability that  $N$  takes each of its possible values?

A We have

$$\mathbb{P}(N = 0) = \mathbb{P}(TT) = 0.2025$$

# Discrete random variables

## Example (Flipping a biased coin twice)

$$\Omega := \{HH, HT, TH, TT\}$$

$$\mathbb{P}(HH) = 0.3025 \quad \mathbb{P}(HT) = 0.2475 \quad \mathbb{P}(TH) = 0.2475 \quad \mathbb{P}(TT) = 0.2025$$

Consider the random variable:

$$N := \# \text{ of heads}$$

| outcome | HH | HT | TH | TT |
|---------|----|----|----|----|
| $N$     | 2  | 1  | 1  | 0  |

Q What is the probability that  $N$  takes each of its possible values?

A We have

$$\mathbb{P}(N = 0) = \mathbb{P}(TT) = 0.2025$$

$$\begin{aligned}\mathbb{P}(N = 1) &= \mathbb{P}(HT) + \mathbb{P}(TH) \\ &= 0.2475 + 0.2475 = 0.4950\end{aligned}$$

# Discrete random variables

## Example (Flipping a biased coin twice)

$$\Omega := \{HH, HT, TH, TT\}$$

$$\mathbb{P}(HH) = 0.3025 \quad \mathbb{P}(HT) = 0.2475 \quad \mathbb{P}(TH) = 0.2475 \quad \mathbb{P}(TT) = 0.2025$$

Consider the random variable:

$$N := \# \text{ of heads}$$

| outcome | HH | HT | TH | TT |
|---------|----|----|----|----|
| $N$     | 2  | 1  | 1  | 0  |

Q What is the probability that  $N$  takes each of its possible values?

A We have

$$\mathbb{P}(N = 0) = \mathbb{P}(TT) = 0.2025$$

$$\begin{aligned}\mathbb{P}(N = 1) &= \mathbb{P}(HT) + \mathbb{P}(TH) \\ &= 0.2475 + 0.2475 = 0.4950\end{aligned}$$

$$\mathbb{P}(N = 2) = \mathbb{P}(HH) = 0.3025$$

# Discrete random variables

## Example (Flipping a biased coin twice)

$$\Omega := \{HH, HT, TH, TT\}$$

$$\mathbb{P}(HH) = 0.3025 \quad \mathbb{P}(HT) = 0.2475 \quad \mathbb{P}(TH) = 0.2475 \quad \mathbb{P}(TT) = 0.2025$$

Consider the random variable:

$$N := \# \text{ of heads}$$

| outcome | HH | HT | TH | TT |
|---------|----|----|----|----|
| $N$     | 2  | 1  | 1  | 0  |

Q What is the probability that  $N$  takes each of its possible values?

A We have

$$\mathbb{P}(N = 0) = \mathbb{P}(TT) = 0.2025$$

$$\begin{aligned}\mathbb{P}(N = 1) &= \mathbb{P}(HT) + \mathbb{P}(TH) \\ &= 0.2475 + 0.2475 = 0.4950\end{aligned}$$

$$\mathbb{P}(N = 2) = \mathbb{P}(HH) = 0.3025$$

The function  $p(k) := \mathbb{P}(N = k)$  is called the **probability distribution** of  $N$ .

# Discrete random variables

## Example (Flipping a biased coin twice)

$$\Omega := \{HH, HT, TH, TT\}$$

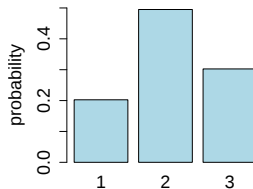
$$\mathbb{P}(HH) = 0.3025 \quad \mathbb{P}(HT) = 0.2475 \quad \mathbb{P}(TH) = 0.2475 \quad \mathbb{P}(TT) = 0.2025$$

$N := \#$  of heads

| outcome | HH | HT | TH | TT |
|---------|----|----|----|----|
| $N$     | 2  | 1  | 1  | 0  |

| $k$ | $\mathbb{P}(N = k)$ |
|-----|---------------------|
| 0   | 0.2025              |
| 1   | 0.4950              |
| 2   | 0.3025              |

Distribution of  $N$



Distribution of  $N$

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{ \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}, \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}, \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}, \dots, \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array} \}$$

$$\mathbb{P}(\begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}) = \mathbb{P}(\begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}) = \mathbb{P}(\begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}) = \dots = \mathbb{P}(\begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}) = 1/36$$

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\square, \square\square, \dots, \square\square\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\square) = \mathbb{P}(\square\square) = \dots = \mathbb{P}(\square\square) = 1/36$$

$X :=$  sum of the numbers on the dice

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{ \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}, \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}, \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}, \dots, \begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array} \}$$

$$\mathbb{P}(\begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}) = \mathbb{P}(\begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}) = \mathbb{P}(\begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}) = \dots = \mathbb{P}(\begin{array}{|c|c|} \hline \cdot & \cdot \\ \hline \end{array}) = 1/36$$

$X :=$  sum of the numbers on the dice

- Q What is the probability that  $X$  takes each of its possible values?

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\bullet, \square\bullet\bullet, \dots, \bullet\bullet\bullet\bullet\bullet\bullet\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\bullet) = \mathbb{P}(\square\bullet\bullet) = \dots = \mathbb{P}(\bullet\bullet\bullet\bullet\bullet\bullet) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X:$

|                  |   |   |   |    |    |    |
|------------------|---|---|---|----|----|----|
|                  | 7 | 8 | 9 | 10 | 11 | 12 |
|                  | 6 | 7 | 8 | 9  | 10 | 11 |
|                  | 5 | 6 | 7 | 8  | 9  | 10 |
|                  | 4 | 5 | 6 | 7  | 8  | 9  |
|                  | 3 | 4 | 5 | 6  | 7  | 8  |
|                  | 2 | 3 | 4 | 5  | 6  | 7  |
| die #2<br>die #1 |   |   |   |    |    |    |

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\cdot, \square\cdot\cdot, \dots, \cdot\cdot\cdot\cdot\cdot\cdot\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\cdot) = \mathbb{P}(\square\cdot\cdot) = \dots = \mathbb{P}(\cdot\cdot\cdot\cdot\cdot\cdot) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X:$

|                 |   |   |   |    |    |    |
|-----------------|---|---|---|----|----|----|
|                 | 7 | 8 | 9 | 10 | 11 | 12 |
|                 | 6 | 7 | 8 | 9  | 10 | 11 |
|                 | 5 | 6 | 7 | 8  | 9  | 10 |
|                 | 4 | 5 | 6 | 7  | 8  | 9  |
|                 | 3 | 4 | 5 | 6  | 7  | 8  |
|                 | 2 | 3 | 4 | 5  | 6  | 7  |
| die #2 \ die #1 |   |   |   |    |    |    |

$$\mathbb{P}(X = 2) = 1/36$$

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\cdot, \square\cdot\cdot, \dots, \blacksquare\blacksquare\blacksquare\blacksquare\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\cdot) = \mathbb{P}(\square\cdot\cdot) = \dots = \mathbb{P}(\blacksquare\blacksquare\blacksquare\blacksquare) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X:$

|                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |
|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|
|  | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                | 12                                                                                |
|  | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                |
|  | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                |
|  | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 |
|  | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 |
|  | 2                                                                                 | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 |
| die #2 \ die #1                                                                   |  |  |  |  |  |  |

$$\mathbb{P}(X = 2) = 1/36$$

$$\mathbb{P}(X = 3) = 2 \times (1/36) = 2/36$$

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\cdot, \square\cdot\cdot, \dots, \blacksquare\blacksquare\blacksquare\blacksquare\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\cdot) = \mathbb{P}(\square\cdot\cdot) = \dots = \mathbb{P}(\blacksquare\blacksquare\blacksquare\blacksquare) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X:$

|                 |   |   |   |    |    |    |
|-----------------|---|---|---|----|----|----|
|                 | 7 | 8 | 9 | 10 | 11 | 12 |
|                 | 6 | 7 | 8 | 9  | 10 | 11 |
|                 | 5 | 6 | 7 | 8  | 9  | 10 |
|                 | 4 | 5 | 6 | 7  | 8  | 9  |
|                 | 3 | 4 | 5 | 6  | 7  | 8  |
|                 | 2 | 3 | 4 | 5  | 6  | 7  |
| die #2 \ die #1 |   |   |   |    |    |    |

$$\mathbb{P}(X = 2) = 1/36$$

$$\mathbb{P}(X = 3) = 2 \times (1/36) = 2/36$$

$$\mathbb{P}(X = 4) = 3 \times (1/36) = 3/36$$

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\cdot, \square\cdot\cdot, \dots, \blacksquare\blacksquare\blacksquare\blacksquare\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\cdot) = \mathbb{P}(\square\cdot\cdot) = \dots = \mathbb{P}(\blacksquare\blacksquare\blacksquare\blacksquare) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X:$

|                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |
|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|
|  | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                | 12                                                                                |
|  | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                |
|  | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                |
|  | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 |
|  | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 |
|  | 2                                                                                 | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 |
| die #2 \ die #1                                                                   |  |  |  |  |  |  |

$$\mathbb{P}(X = 2) = 1/36$$

$$\mathbb{P}(X = 3) = 2 \times (1/36) = 2/36$$

$$\mathbb{P}(X = 4) = 3 \times (1/36) = 3/36$$

$$\mathbb{P}(X = 5) = 4 \times (1/36) = 4/36$$

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\cdot, \square\cdot\cdot, \dots, \blacksquare\blacksquare\blacksquare\blacksquare\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\cdot) = \mathbb{P}(\square\cdot\cdot) = \dots = \mathbb{P}(\blacksquare\blacksquare\blacksquare\blacksquare) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X:$

|  | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                | 12                                                                                |
|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|
|  | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                |
|  | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                |
|  | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 |
|  | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 |
|  | 2                                                                                 | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 |
| die #1                                                                            |  |  |  |  |  |  |

$$\mathbb{P}(X = 2) = 1/36$$

$$\mathbb{P}(X = 3) = 2 \times (1/36) = 2/36$$

$$\mathbb{P}(X = 4) = 3 \times (1/36) = 3/36$$

$$\mathbb{P}(X = 5) = 4 \times (1/36) = 4/36$$

$$\mathbb{P}(X = 6) = 5 \times (1/36) = 5/36$$

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\cdot, \square\cdot\cdot, \dots, \blacksquare\blacksquare\blacksquare\blacksquare\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\cdot) = \mathbb{P}(\square\cdot\cdot) = \dots = \mathbb{P}(\blacksquare\blacksquare\blacksquare\blacksquare) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X:$

|                 |   |   |   |    |    |    |
|-----------------|---|---|---|----|----|----|
|                 | 7 | 8 | 9 | 10 | 11 | 12 |
|                 | 6 | 7 | 8 | 9  | 10 | 11 |
|                 | 5 | 6 | 7 | 8  | 9  | 10 |
|                 | 4 | 5 | 6 | 7  | 8  | 9  |
|                 | 3 | 4 | 5 | 6  | 7  | 8  |
|                 | 2 | 3 | 4 | 5  | 6  | 7  |
| die #2 \ die #1 |   |   |   |    |    |    |

$$\mathbb{P}(X = 2) = 1/36$$

$$\mathbb{P}(X = 3) = 2 \times (1/36) = 2/36$$

$$\mathbb{P}(X = 4) = 3 \times (1/36) = 3/36$$

$$\mathbb{P}(X = 5) = 4 \times (1/36) = 4/36$$

$$\mathbb{P}(X = 6) = 5 \times (1/36) = 5/36$$

$$\mathbb{P}(X = 7) = 6 \times (1/36) = 6/36$$

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\cdot, \square\cdot\cdot, \dots, \blacksquare\blacksquare\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\cdot) = \mathbb{P}(\square\cdot\cdot) = \dots = \mathbb{P}(\blacksquare\blacksquare) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X:$

|                 |   |   |   |    |    |    |
|-----------------|---|---|---|----|----|----|
|                 | 7 | 8 | 9 | 10 | 11 | 12 |
|                 | 6 | 7 | 8 | 9  | 10 | 11 |
|                 | 5 | 6 | 7 | 8  | 9  | 10 |
|                 | 4 | 5 | 6 | 7  | 8  | 9  |
|                 | 3 | 4 | 5 | 6  | 7  | 8  |
|                 | 2 | 3 | 4 | 5  | 6  | 7  |
| die #2 \ die #1 |   |   |   |    |    |    |

$$\mathbb{P}(X = 2) = 1/36$$

$$\mathbb{P}(X = 3) = 2 \times (1/36) = 2/36$$

$$\mathbb{P}(X = 4) = 3 \times (1/36) = 3/36$$

$$\mathbb{P}(X = 5) = 4 \times (1/36) = 4/36$$

$$\mathbb{P}(X = 6) = 5 \times (1/36) = 5/36$$

$$\mathbb{P}(X = 7) = 6 \times (1/36) = 6/36$$

$$\mathbb{P}(X = 8) = 5 \times (1/36) = 5/36$$

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\cdot, \square\cdot\cdot, \dots, \blacksquare\blacksquare\blacksquare\blacksquare\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\cdot) = \mathbb{P}(\square\cdot\cdot) = \dots = \mathbb{P}(\blacksquare\blacksquare\blacksquare\blacksquare) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X$ :

|                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |
|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|
|  | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                | 12                                                                                |
|  | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                |
|  | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                |
|  | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 |
|  | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 |
|  | 2                                                                                 | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 |
| die #2 \ die #1                                                                   |  |  |  |  |  |  |

$$\mathbb{P}(X = 2) = 1/36$$

$$\mathbb{P}(X = 3) = 2 \times (1/36) = 2/36$$

$$\mathbb{P}(X = 4) = 3 \times (1/36) = 3/36$$

$$\mathbb{P}(X = 5) = 4 \times (1/36) = 4/36$$

$$\mathbb{P}(X = 6) = 5 \times (1/36) = 5/36$$

$$\mathbb{P}(X = 7) = 6 \times (1/36) = 6/36$$

$$\mathbb{P}(X = 8) = 5 \times (1/36) = 5/36$$

$$\mathbb{P}(X = 9) = 4 \times (1/36) = 4/36$$

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\cdot, \square\cdot\cdot, \dots, \blacksquare\blacksquare\blacksquare\blacksquare\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\cdot) = \mathbb{P}(\square\cdot\cdot) = \dots = \mathbb{P}(\blacksquare\blacksquare\blacksquare\blacksquare) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X:$

|                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |
|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|
|  | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                | 12                                                                                |
|  | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                |
|  | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                |
|  | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 |
|  | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 |
|  | 2                                                                                 | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 |
| die #2<br>die #1                                                                  |  |  |  |  |  |  |

$$\mathbb{P}(X = 2) = 1/36$$

$$\mathbb{P}(X = 3) = 2 \times (1/36) = 2/36$$

$$\mathbb{P}(X = 4) = 3 \times (1/36) = 3/36$$

$$\mathbb{P}(X = 5) = 4 \times (1/36) = 4/36$$

$$\mathbb{P}(X = 6) = 5 \times (1/36) = 5/36$$

$$\mathbb{P}(X = 7) = 6 \times (1/36) = 6/36$$

$$\mathbb{P}(X = 8) = 5 \times (1/36) = 5/36$$

$$\mathbb{P}(X = 9) = 4 \times (1/36) = 4/36$$

$$\mathbb{P}(X = 10) = 3 \times (1/36) = 3/36$$

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\bullet, \square\bullet\bullet, \dots, \blacksquare\blacksquare\blacksquare\blacksquare\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\bullet) = \mathbb{P}(\square\bullet\bullet) = \dots = \mathbb{P}(\blacksquare\blacksquare\blacksquare\blacksquare) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X$ :

|                  |   |   |   |    |    |    |
|------------------|---|---|---|----|----|----|
|                  | 7 | 8 | 9 | 10 | 11 | 12 |
|                  | 6 | 7 | 8 | 9  | 10 | 11 |
|                  | 5 | 6 | 7 | 8  | 9  | 10 |
|                  | 4 | 5 | 6 | 7  | 8  | 9  |
|                  | 3 | 4 | 5 | 6  | 7  | 8  |
|                  | 2 | 3 | 4 | 5  | 6  | 7  |
| die #2<br>die #1 |   |   |   |    |    |    |

$$\mathbb{P}(X = 2) = 1/36$$

$$\mathbb{P}(X = 3) = 2 \times (1/36) = 2/36$$

$$\mathbb{P}(X = 4) = 3 \times (1/36) = 3/36$$

$$\mathbb{P}(X = 5) = 4 \times (1/36) = 4/36$$

$$\mathbb{P}(X = 6) = 5 \times (1/36) = 5/36$$

$$\mathbb{P}(X = 7) = 6 \times (1/36) = 6/36$$

$$\mathbb{P}(X = 8) = 5 \times (1/36) = 5/36$$

$$\mathbb{P}(X = 9) = 4 \times (1/36) = 4/36$$

$$\mathbb{P}(X = 10) = 3 \times (1/36) = 3/36$$

$$\mathbb{P}(X = 11) = 2 \times (1/36) = 2/36$$

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\cdot, \cdot\square, \cdot\cdot, \dots, \blacksquare\blacksquare\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\cdot) = \mathbb{P}(\cdot\square) = \dots = \mathbb{P}(\blacksquare\blacksquare) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X$ :

|                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |
|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|
|  | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                | 12                                                                                |
|  | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                |
|  | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                |
|  | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 |
|  | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 |
|  | 2                                                                                 | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 |
| die #2 \ die #1                                                                   |  |  |  |  |  |  |

$$\mathbb{P}(X = 2) = 1/36$$

$$\mathbb{P}(X = 3) = 2 \times (1/36) = 2/36$$

$$\mathbb{P}(X = 4) = 3 \times (1/36) = 3/36$$

$$\mathbb{P}(X = 5) = 4 \times (1/36) = 4/36$$

$$\mathbb{P}(X = 6) = 5 \times (1/36) = 5/36$$

$$\mathbb{P}(X = 7) = 6 \times (1/36) = 6/36$$

$$\mathbb{P}(X = 8) = 5 \times (1/36) = 5/36$$

$$\mathbb{P}(X = 9) = 4 \times (1/36) = 4/36$$

$$\mathbb{P}(X = 10) = 3 \times (1/36) = 3/36$$

$$\mathbb{P}(X = 11) = 2 \times (1/36) = 2/36$$

$$\mathbb{P}(X = 12) = 1/36$$

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\cdot, \square\cdot\cdot, \dots, \cdot\cdot\cdot\cdot\cdot\cdot\}$$

$$\mathbb{P}(\square\square) = \mathbb{P}(\square\cdot) = \mathbb{P}(\square\cdot\cdot) = \dots = \mathbb{P}(\cdot\cdot\cdot\cdot\cdot\cdot) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X$ :

|                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |
|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|
|  | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                | 12                                                                                |
|  | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                |
|  | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                |
|  | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 |
|  | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 |
|  | 2                                                                                 | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 |
| die #2<br>die #1                                                                  |  |  |  |  |  |  |

| $a$ | $\mathbb{P}(X = a)$ |
|-----|---------------------|
| 2   | 1/36                |
| 3   | 2/36                |
| 4   | 3/36                |
| 5   | 4/36                |
| 6   | 5/36                |
| 7   | 6/36                |
| 8   | 5/36                |
| 9   | 4/36                |
| 10  | 3/36                |
| 11  | 2/36                |
| 12  | 1/36                |

# Discrete random variables

## Example (Rolling two dice)

$$\Omega := \{\square\square, \square\cdot, \square\cdot\cdot, \dots, \cdot\cdot\cdot\cdot\cdot\cdot\}$$

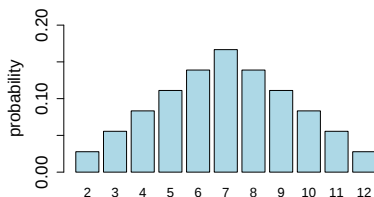
$$\mathbb{P}(\square\square) = \mathbb{P}(\square\cdot) = \mathbb{P}(\square\cdot\cdot) = \dots = \mathbb{P}(\cdot\cdot\cdot\cdot\cdot\cdot) = 1/36$$

$X :=$  sum of the numbers on the dice

Q What is the probability that  $X$  takes each of its possible values?

$X$ :

|                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |                                                                                   |
|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|
|  | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                | 12                                                                                |
|  | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                | 11                                                                                |
|  | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 | 10                                                                                |
|  | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 | 9                                                                                 |
|  | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 | 8                                                                                 |
|  | 2                                                                                 | 3                                                                                 | 4                                                                                 | 5                                                                                 | 6                                                                                 | 7                                                                                 |
| die #2 \ die #1                                                                   |  |  |  |  |  |  |



# Distribution of discrete random variables

The **probability distribution** (a.k.a. the **probability mass function**) of a discrete RV  $X$  is the function  $p(a) := \mathbb{P}(X = a)$  that assigns, to each possible value of  $X$ , the probability that  $X$  takes that value.

The information in the distribution of  $X$  can be summarized in a table, a bar chart, or a formula.

# Distribution of discrete random variables

The **probability distribution** (a.k.a. the **probability mass function**) of a discrete RV  $X$  is the function  $p(a) := \mathbb{P}(X = a)$  that assigns, to each possible value of  $X$ , the probability that  $X$  takes that value.

The information in the distribution of  $X$  can be summarized in a table, a bar chart, or a formula.

## Properties of probability distribution

The distribution of every discrete RV  $X$  satisfies the following:

- For each value  $a$ ,

$$0 \leq p(a) \leq 1 .$$

- If  $X$  has possible values  $a_1, a_2, a_3, \dots$ , then

$$p(a_1) + p(a_2) + p(a_3) + \dots = 1 .$$