American University of Beirut
STAT 210: Elementary Statistics for Sciences
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Siamak Taati

Chapter 4
Probability
Part 2

Probability

Probability model of a random experiment

Terminology

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- Any event concerning the experiment can be identified by the collection of outcomes that realize that event.

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Example (Rolling a die)

- The sample space is $\Omega := \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}$.
- An (non-elementary) event: $A := \{⚁, ⚃, ⚅\}$
- An (elementary) event: $B := \{⚄\}$.

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  - Two trivial events:

    $\varnothing := \{\}$ (<u>nothing</u> happens)

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- An (non-elementary) event: $A :=$ {⚁, ⚃, ⚅}
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Probability model of a random experiment

Terminology

Suppose we want to talk about a particular random experiment.

- The collection of all possible **outcomes** of the experiment is called the **sample space** and is often denoted by Ω.
- Any **event** concerning the experiment can be identified by the collection of outcomes that **realize** that event.
  - An **elementary** event is an event which is realized by exactly one possible outcome.
  - Two **trivial** events:

    $\varnothing := \{\}$ (<u>nothing</u> happens) and Ω (<u>something</u> happens)

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Probability model of a random experiment

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- The probability of an event A is a number $\mathbb{P}(A)$ between 0 and 1, which indicates how likely it is for that event to occur.

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Example (Rolling a die)

- The sample space is $\Omega := \{$⚀, ⚁, ⚂, ⚃, ⚄, ⚅$\}$.
- $\mathbb{P}(\{$⚁, ⚃, ⚅$\}) = {}^{1}\!/_{2}$.

Probability model of a random experiment

Terminology

- The probability of an event A is a number $\mathbb{P}(A)$ between 0 and 1, which indicates how likely it is for that event to occur.
- In particular, if a is a possible outcome, we write $\mathbb{P}(a)$ for the probability that a becomes the actual outcome of the experiment.

Example (Rolling a die)

- The sample space is $\Omega := \{\epsdice{1}, \epsdice{2}, \epsdice{3}, \epsdice{4}, \epsdice{5}, \epsdice{6}\}$.
- $\mathbb{P}(\{\epsdice{2}, \epsdice{4}, \epsdice{6}\}) = 1/2$.

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Example (Rolling a die)

- The sample space is $\Omega :=$ {⚀, ⚁, ⚂, ⚃, ⚄, ⚅}.
- $\mathbb{P}($ {⚁, ⚃, ⚅} $) = 1/2$.
- $\mathbb{P}($⚄$) = 1/6$.

Probability model of a random experiment

Consistency of probabilities

Example (Rolling a die)

- The sample space is $\Omega :=$ {⚀, ⚁, ⚂, ⚃, ⚄, ⚅}.

Probability model of a random experiment

Consistency of probabilities

- If the sample space is $\Omega = \{c_1, c_2, c_3, \ldots\}$, then

$$\boxed{\mathbb{P}(c_1) + \mathbb{P}(c_2) + \mathbb{P}(c_3) + \cdots = 1\ .}$$

(In words: The probabilities of the possible outcomes add up to 1.)

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- $\mathbb{P}(\text{⚀}) + \mathbb{P}(\text{⚁}) + \mathbb{P}(\text{⚂}) + \mathbb{P}(\text{⚃}) + \mathbb{P}(\text{⚄}) + \mathbb{P}(\text{⚅}) = 1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6 = 1.$

Probability model of a random experiment

Consistency of probabilities

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$$\boxed{\mathbb{P}(c_1) + \mathbb{P}(c_2) + \mathbb{P}(c_3) + \cdots = 1\ .}$$

(In words: The probabilities of the possible outcomes add up to 1.)

- For every event $A = \{a_1, a_2, a_3, \ldots\}$,

$$\boxed{\mathbb{P}(A) = \mathbb{P}(a_1) + \mathbb{P}(a_2) + \mathbb{P}(a_3) + \cdots}$$

(In words: The probability of an event is the sum of the probabilities of the possible outcomes that realize that event.)

Example (Rolling a die)

- The sample space is $\Omega := \{$⚀, ⚁, ⚂, ⚃, ⚄, ⚅$\}$.
- $\mathbb{P}($⚀$)+\mathbb{P}($⚁$)+\mathbb{P}($⚂$)+\mathbb{P}($⚃$)+\mathbb{P}($⚄$)+\mathbb{P}($⚅$) = {}^1/_6+{}^1/_6+{}^1/_6+{}^1/_6+{}^1/_6+{}^1/_6 = 1.$

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- $\mathbb{P}(\{\text{⚁}, \text{⚃}, \text{⚅}\}) = \mathbb{P}(\text{⚁}) + \mathbb{P}(\text{⚃}) + \mathbb{P}(\text{⚅}) = 1/6 + 1/6 + 1/6 = 1/2.$

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- $\mathbb{P}(\varnothing) = 0$. (The probability that nothing happens is zero. Duh!)

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  (In words: The probability of an event is the sum of the probabilities of the possible outcomes that realize that event.)

- $\mathbb{P}(\varnothing) = 0$. (The probability that nothing happens is zero. Duh!)

- $\mathbb{P}(\Omega) = 1$. (With probability 1, something happens. Duh!)

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$$\boxed{\mathbb{P}(A) = \mathbb{P}(a_1) + \mathbb{P}(a_2) + \mathbb{P}(a_3) + \cdots}$$

  (In words: The probability of an event is the sum of the probabilities of the possible outcomes that realize that event.)

- $\mathbb{P}(\varnothing) = 0$. (The probability that nothing happens is zero. Duh!)
  Terminology: A null event is an event which occurs with probability 0.

- $\mathbb{P}(\Omega) = 1$. (With probability 1, something happens. Duh!)
  Terminology: A sure event is an event which occurs with probability 1.

Constructing a probability model

Constructing a probability model for a random experiment involves assigning a probability $\mathbb{P}(A)$ to each event A in such a way that the probabilities of different events are consistent.

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A In general, verifying consistency can be rather complicated. However, . . .

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- $\longrightarrow$ Based on symmetry
- $\longrightarrow$ Probabilities as idealized frequencies
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Q How can we ensure the consistency of the probabilities?

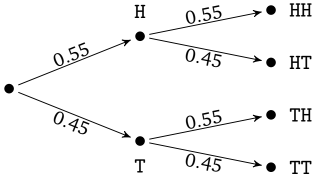
A In general, verifying consistency can be rather complicated. However, for experiments that are "simple enough", we only need to choose the probabilities of individual outcomes in a consistent way.

Working with a probability model

Working with a probability model

Example (Flipping a biased coin twice)

$$\boxed{\Omega := \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}}$$



$$\mathbb{P}(\mathtt{HH}) = 0.55 \times 0.55 = 0.3025$$
$$\mathbb{P}(\mathtt{HT}) = 0.55 \times 0.45 = 0.2475$$
$$\mathbb{P}(\mathtt{TH}) = 0.45 \times 0.55 = 0.2475$$
$$\mathbb{P}(\mathtt{TT}) = 0.45 \times 0.45 = 0.2025$$

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Example (Flipping a biased coin twice)

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Consider the following event:

$$A := \langle \text{both flips show same side} \rangle$$

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Consider the following event:

$$A := \langle \text{both flips show same side} \rangle = \{\mathtt{HH}, \mathtt{TT}\}$$

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Consider the following event:

$$A := \langle \text{both flips show same side} \rangle = \{\mathtt{HH}, \mathtt{TT}\}$$

We earlier saw that

$$\mathbb{P}(A) = 0.5050$$

Working with a probability model

Example (Flipping a biased coin twice)

$$\boxed{\Omega := \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}}$$

$$\boxed{\mathbb{P}(\mathtt{HH}) = 0.3025 \qquad \mathbb{P}(\mathtt{HT}) = 0.2475 \qquad \mathbb{P}(\mathtt{TH}) = 0.2475 \qquad \mathbb{P}(\mathtt{TT}) = 0.2025}$$

Consider the following event:

$$A := \langle \text{both flips show same side} \rangle = \{\mathtt{HH}, \mathtt{TT}\}$$

We earlier saw that

$$\mathbb{P}(A) = 0.5050$$

Q What is the probability that $\underline{A \text{ does not happen}}$?

Working with a probability model

Example (Flipping a biased coin twice)

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Consider the following event:

$$A := \langle \text{both flips show same side} \rangle = \{\mathtt{HH}, \mathtt{TT}\}$$

We earlier saw that

$$\mathbb{P}(A) = 0.5050$$

Q What is the probability that $\underline{A \text{ does not happen}}$?

A1 $\langle A$ does not happen$\rangle = \{\mathtt{HT}, \mathtt{TH}\}$. Hence,

$$\mathbb{P}(\text{not } A) = \mathbb{P}(\mathtt{HT}) + \mathbb{P}(\mathtt{TH}) = 0.2475 + 0.2475 = \boxed{0.4950}$$

Working with a probability model

Example (Flipping a biased coin twice)

$$\boxed{\Omega := \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}}$$

$$\boxed{\mathbb{P}(\mathtt{HH}) = 0.3025 \qquad \mathbb{P}(\mathtt{HT}) = 0.2475 \qquad \mathbb{P}(\mathtt{TH}) = 0.2475 \qquad \mathbb{P}(\mathtt{TT}) = 0.2025}$$

Consider the following event:

$$A := \langle \text{both flips show same side} \rangle = \{\mathtt{HH}, \mathtt{TT}\}$$

We earlier saw that

$$\mathbb{P}(A) = 0.5050$$

Q What is the probability that <u>A does not happen</u>?

A2 The probabilities of possible outcomes must add up to 1. Hence,

$$\mathbb{P}(\text{not } A) = 1 - \mathbb{P}(A) = 1 - 0.5050 = \boxed{0.4950}$$

Working with a probability model

Example (Flipping a biased coin twice)

$$\boxed{\Omega := \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}}$$

$$\boxed{\mathbb{P}(\mathtt{HH}) = 0.3025 \qquad \mathbb{P}(\mathtt{HT}) = 0.2475 \qquad \mathbb{P}(\mathtt{TH}) = 0.2475 \qquad \mathbb{P}(\mathtt{TT}) = 0.2025}$$

Consider the following event:

$$A := \langle \text{both flips show same side} \rangle = \{\mathtt{HH}, \mathtt{TT}\}$$

We earlier saw that

$$\mathbb{P}(A) = 0.5050$$

Q What is the probability that $\underline{A \text{ does not happen}}$?

A2 The probabilities of possible outcomes must add up to 1. Hence,

$$\mathbb{P}(\text{not } A) = 1 - \mathbb{P}(A) = 1 - 0.5050 = \boxed{0.4950}$$

The event $\langle \text{not } A \rangle$ is called the complement of A.

Working with a probability model

Example (Flipping a biased coin twice)

$$\boxed{\Omega := \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}}$$

$$\boxed{\mathbb{P}(\mathtt{HH}) = 0.3025 \qquad \mathbb{P}(\mathtt{HT}) = 0.2475 \qquad \mathbb{P}(\mathtt{TH}) = 0.2475 \qquad \mathbb{P}(\mathtt{TT}) = 0.2025}$$

Consider the following two events:

$$A := \langle \text{both flips show same side} \rangle = \{\mathtt{HH}, \mathtt{TT}\}$$
$$C := \langle \text{2nd flip comes up heads} \rangle = \{\mathtt{HH}, \mathtt{TH}\}$$

We earlier saw that

$$\mathbb{P}(A) = 0.5050 \qquad\qquad \mathbb{P}(C) = 0.5500$$

Working with a probability model

Example (Flipping a biased coin twice)

$$\boxed{\Omega := \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}}$$

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Consider the following two events:

$$A := \langle \text{both flips show same side} \rangle = \{\mathtt{HH}, \mathtt{TT}\}$$

$$C := \langle \text{2nd flip comes up heads} \rangle = \{\mathtt{HH}, \mathtt{TH}\}$$

We earlier saw that

$$\mathbb{P}(A) = 0.5050 \qquad\qquad \mathbb{P}(C) = 0.5500$$

Q What is the probability that <u>both A and C happen</u>?

Working with a probability model

Example (Flipping a biased coin twice)

$$\boxed{\Omega := \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}}$$

$$\boxed{\mathbb{P}(\mathtt{HH}) = 0.3025 \qquad \mathbb{P}(\mathtt{HT}) = 0.2475 \qquad \mathbb{P}(\mathtt{TH}) = 0.2475 \qquad \mathbb{P}(\mathtt{TT}) = 0.2025}$$

Consider the following two events:

$$A := \langle \text{both flips show same side} \rangle = \{\mathtt{HH}, \mathtt{TT}\}$$
$$C := \langle \text{2nd flip comes up heads} \rangle = \{\mathtt{HH}, \mathtt{TH}\}$$

We earlier saw that

$$\mathbb{P}(A) = 0.5050 \qquad \mathbb{P}(C) = 0.5500$$

Q What is the probability that both A and C happen?

A $\langle \text{both } A \text{ and } C \text{ happen} \rangle = \{\mathtt{HH}\}$. Hence,

$$\mathbb{P}(A \text{ and } C) = \mathbb{P}(\mathtt{HH}) = \boxed{0.3025}$$

Working with a probability model

Example (Flipping a biased coin twice)

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Consider the following two events:

$$A := \langle \text{both flips show same side} \rangle = \{\mathtt{HH}, \mathtt{TT}\}$$
$$C := \langle \text{2nd flip comes up heads} \rangle = \{\mathtt{HH}, \mathtt{TH}\}$$

We earlier saw that

$$\mathbb{P}(A) = 0.5050 \qquad \mathbb{P}(C) = 0.5500$$

Q What is the probability that both A and C happen?

A $\langle \text{both } A \text{ and } C \text{ happen} \rangle = \{\mathtt{HH}\}$. Hence,

$$\mathbb{P}(A \text{ and } C) = \mathbb{P}(\mathtt{HH}) = \boxed{0.3025}$$

The event $\langle A \text{ and } C \rangle$ is called the intersection of A and C.

Working with a probability model

Example (Flipping a biased coin twice)

$$\boxed{\Omega := \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}}$$

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Consider the following three events:

$$A := \langle \text{both flips show same side} \rangle = \{\mathtt{HH}, \mathtt{TT}\}$$

$$B := \langle \text{1st flip comes up heads} \rangle = \{\mathtt{HH}, \mathtt{HT}\}$$

$$C := \langle \text{2nd flip comes up heads} \rangle = \{\mathtt{HH}, \mathtt{TH}\}$$

We already know that

$$\mathbb{P}(A) = 0.5050 \qquad \mathbb{P}(B) = \mathbb{P}(C) = 0.5500$$

Working with a probability model

Example (Flipping a biased coin twice)

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$$C := \langle \text{2nd flip comes up heads} \rangle = \{\mathtt{HH}, \mathtt{TH}\}$$

We already know that

$$\mathbb{P}(A) = 0.5050 \qquad \mathbb{P}(B) = \mathbb{P}(C) = 0.5500$$

Q What is the probability that both B and C happen?

Working with a probability model

Example (Flipping a biased coin twice)

$$\boxed{\Omega := \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}}$$

$$\boxed{\mathbb{P}(\mathtt{HH}) = 0.3025 \qquad \mathbb{P}(\mathtt{HT}) = 0.2475 \qquad \mathbb{P}(\mathtt{TH}) = 0.2475 \qquad \mathbb{P}(\mathtt{TT}) = 0.2025}$$

Consider the following three events:

$$A := \langle \text{both flips show same side} \rangle = \{\mathtt{HH}, \mathtt{TT}\}$$
$$B := \langle \text{1st flip comes up heads} \rangle = \{\mathtt{HH}, \mathtt{HT}\}$$
$$C := \langle \text{2nd flip comes up heads} \rangle = \{\mathtt{HH}, \mathtt{TH}\}$$

We already know that

$$\mathbb{P}(A) = 0.5050 \qquad \mathbb{P}(B) = \mathbb{P}(C) = 0.5500$$

Q What is the probability that both B and C happen?

A1 $\langle \text{both } B \text{ and } C \text{ happen} \rangle = \{\mathtt{HH}\}$. Hence,

$$\mathbb{P}(B \text{ and } C) = \mathbb{P}(\mathtt{HH}) = \boxed{0.3025}$$

Working with a probability model

Example (Flipping a biased coin twice)

$$\boxed{\Omega := \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}}$$

$$\boxed{\mathbb{P}(\mathtt{HH}) = 0.3025 \qquad \mathbb{P}(\mathtt{HT}) = 0.2475 \qquad \mathbb{P}(\mathtt{TH}) = 0.2475 \qquad \mathbb{P}(\mathtt{TT}) = 0.2025}$$

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$$C := \langle \text{2nd flip comes up heads} \rangle = \{\mathtt{HH}, \mathtt{TH}\}$$

We already know that

$$\mathbb{P}(A) = 0.5050 \qquad \mathbb{P}(B) = \mathbb{P}(C) = 0.5500$$

Q What is the probability that both B and C happen?

A2 The occurrence of B does not affect C or vice versa. Hence,

$$\mathbb{P}(B \text{ and } C) = \mathbb{P}(B)\,\mathbb{P}(C) = 0.55 \times 0.55 = \boxed{0.3025}$$

Working with a probability model

Example (Flipping a biased coin twice)

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$$\mathbb{P}(B \text{ and } C) = \mathbb{P}(B)\,\mathbb{P}(C) = 0.55 \times 0.55 = \boxed{0.3025}$$

We say that B and C are independent.

Working with a probability model

Example (Flipping a biased coin twice)

$$\boxed{\Omega := \{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}}$$

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$$A := \langle \text{both flips show same side} \rangle = \{\mathtt{HH}, \mathtt{TT}\}$$
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We already know that

$$\mathbb{P}(A) = 0.5050 \qquad \mathbb{P}(C) = 0.5500 \qquad \mathbb{P}(A \text{ and } C) = 0.3025$$

Working with a probability model

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$$\mathbb{P}(A) = 0.5050 \qquad \mathbb{P}(C) = 0.5500 \qquad \mathbb{P}(A \text{ and } C) = 0.3025$$

Q What is the probability that <u>either A or C (or both) happens</u>?

Working with a probability model

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Q What is the probability that either A or C (or both) happens?

$\boxed{\text{A1}}$ $\langle$either A or C happens$\rangle = \{\mathtt{HH}, \mathtt{TH}, \mathtt{TT}\}$. Hence,

$$\mathbb{P}(A \text{ or } C) = \mathbb{P}(\mathtt{HH}) + \mathbb{P}(\mathtt{TH}) + \mathbb{P}(\mathtt{TT}) = 0.3025 + 0.2475 + 0.2025 = \boxed{0.7525}$$

Working with a probability model

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$\boxed{\text{A1}}$ $\langle$either A or C happens$\rangle = \{\mathtt{HH}, \mathtt{TH}, \mathtt{TT}\}$. Hence,

$$\mathbb{P}(A \text{ or } C) = \mathbb{P}(\mathtt{HH}) + \mathbb{P}(\mathtt{TH}) + \mathbb{P}(\mathtt{TT}) = 0.3025 + 0.2475 + 0.2025 = \boxed{0.7525}$$

The event $\langle A \text{ or } C \rangle$ is called the union of A and C.

Working with a probability model

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$$\mathbb{P}(A) = 0.5050 \qquad \mathbb{P}(C) = 0.5500 \qquad \mathbb{P}(A \text{ and } C) = 0.3025$$

Q What is the probability that <u>either A or C (or both) happens</u>?

A2 $\langle$neither A nor C happens$\rangle = \{\mathtt{HT}\}$. Hence,

$$\mathbb{P}(A \text{ or } C) = 1 - \mathbb{P}(\text{neither } A \text{ nor } C) = 1 - \mathbb{P}(\mathtt{HT})$$
$$= 1 - 0.2475 = \boxed{0.7525}$$

Working with a probability model

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We already know that

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Q What is the probability that either A or C (or both) happens?

A3 Note the double counting:

$$\mathbb{P}(A \text{ or } C) = \mathbb{P}(A) + \mathbb{P}(C) - \mathbb{P}(A \text{ and } C)$$
$$= 0.5050 + 0.5500 - 0.3025 = \boxed{0.7525}$$

Working with a probability model

Events described in terms of other events

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- The intersection of two events A and B is the event that both A and B occur. It is denoted by $\langle A \text{ and } B \rangle$ or $A \cap B$.

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General relations

Working with a probability model

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General relations

I. For every event A,

$$\boxed{\mathbb{P}(\text{not } A) = 1 - \mathbb{P}(A)}$$

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General relations

I. For every event A,

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II. For every two events A and B, [The inclusion-exclusion principle]

$$\boxed{\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \text{ and } B)}$$

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$$\boxed{\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \text{ and } B)} \leq \mathbb{P}(A) + \mathbb{P}(B)$$

Working with a probability model

Mutually exclusive events

Two events A and B are said to be mutually exclusive (or disjoint) if they cannot occur simultaneously,
that is, if $\langle A \text{ and } B \rangle = \varnothing$ (no outcome realizes both A and B).

Independent events

Two events A and B are said to be independent if the occurrence of one does not affect the occurrence of the other.

Working with a probability model

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$$\boxed{\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B)}$$

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Two events A and B are said to be independent if the occurrence of one does not affect the occurrence of the other.

- If $\underline{A \text{ and } B \text{ are independent}}$, then

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