

American University of Beirut
Introduction to Ergodic Theory

MATH 307K (2024–2025 Spring)

Assignment 3

Problem 1 (Proof of Birkhoff's ergodic theorem). Complete the proof of Birkhoff's ergodic theory (i.e., remove the boundedness assumption) by following the approach sketched in the posted notes (or any other approach).

Problem 2 (Induced isometries). Let μ be a probability measure on a measurable space $\mathcal{X}$ and $T: \mathcal{X} \rightarrow \mathcal{X}$ a measurable map that preserves μ . Verify that for every $1 \leq p \leq \infty$, the map $f \mapsto f \circ T$ is a positive linear isometry of $L^p_\mu(\mathcal{X})$.

Problem 3 (Property of conditional expectation). Let $(\mathcal{X}, \mathcal{F})$ be a measurable space and $T: \mathcal{X} \rightarrow \mathcal{X}$ a measurable map. Prove that

$$\mu(f | \mathcal{A}) \circ T = \mu(f \circ T | T^{-1}\mathcal{A}) \quad \mu\text{-a.e.}$$

for every probability measure μ on $\mathcal{X}$, every function $f \in L^1_\mu(\mathcal{X})$ and every sub- σ -algebra $\mathcal{A} \subseteq \mathcal{F}$.

Problem 4 (Limit in Birkhoff's theorem). Prove that the following is equivalent to the original statement of Birkhoff's ergodic theorem discussed in class and the notes:

Theorem. Let $(\mathcal{X}, \mathcal{F}, \mu)$ be a probability space and $T: \mathcal{X} \rightarrow \mathcal{X}$ a measurable map that preserves μ . For every $f \in L^1_\mu(\mathcal{X})$, we have

$$\frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \rightarrow \mu(f | \mathcal{I}_\mu)$$

μ -almost everywhere as $n \rightarrow \infty$, where $\mathcal{I}_\mu$ is the sub- σ -algebra of invariant sets modulo μ .

Hint: For the forward direction, take an arbitrary $E \in \mathcal{I}_\mu$ and apply the original form of Birkhoff's theorem to $f \cdot \mathbb{1}_E$.

Problem 5 (Weak mixing: irrational rotation). Let $\mathcal{X} := \mathbb{R}/\mathbb{Z}$. Let α be an irrational, and let $R_\alpha := x + \alpha \pmod{1}$ be the *rotation-by- α* map on $\mathcal{X}$. Prove that R_α is not weakly mixing.

Problem 6 (Cartesian product: ergodicity and mixing). Recall that the (Cartesian) product of two measure-preserving dynamical systems $(\mathcal{X}_1, \mathcal{F}_1, \mu_1, T_1)$ and $(\mathcal{X}_2, \mathcal{F}_2, \mu_2, T_2)$ is the system

$$(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \times \mu_2, T_1 \times T_2),$$

where $\mathcal{F}_1 \otimes \mathcal{F}_2$ is the product σ -algebra, $\mu_1 \times \mu_2$ is the product measure, and $(T_1 \times T_2)(x_1, x_2) := (T_1(x_1), T_2(x_2))$. We saw that the product of two weakly mixing systems is again weakly mixing.

- Give an example of two ergodic systems the product of which is not ergodic.
- Is the product of two strongly mixing systems strongly mixing?
- Prove that the product of an ergodic system and a weakly mixing system is ergodic.

Problem 7 (Counter-examples). Recall the following proposition, which was proven in class using the “full-density diagonal sequence” lemma:

Proposition. Let $a_0, a_1, \dots$ be a bounded sequence of non-negative real numbers. The following conditions are equivalent:

- (Cesàro convergence) $\frac{1}{n} \sum_{k=0}^{n-1} a_k \rightarrow 0$ as $n \rightarrow \infty$.
- (convergence in density) There exists a full-density set $J \subseteq \mathbb{N}$ such that $a_n \rightarrow 0$ as $J \ni n \rightarrow \infty$.

Provide counter-examples showing that the *boundedness* and *non-negativity* assumptions cannot be removed.

Problem 8 (Convergence in density vs. statistical convergence). Let $(\mathcal{X}, \rho)$ be a metric space. Prove that for every $a_0, a_1, a_2, \dots \in \mathcal{X}$ and $a \in \mathcal{X}$, the following conditions are equivalent:

- (statistical convergence) For every $\varepsilon > 0$, we have $\lim_{n \rightarrow \infty} \frac{|\{k < n : \rho(a_k, a) \geq \varepsilon\}|}{n} = 0$.
- (convergence in density) There exists a set $J \subseteq \mathbb{N}$ of full density such that $\lim_{J \ni n \rightarrow \infty} a_n = a$.