American University of Beirut

Introduction to Ergodic Theory

MATH 307K (2024–2025 Spring)

Assignment 2

Problem 1 (True or False). Let $T\colon \mathcal{X} \to \mathcal{X}$ be a map on a set $\mathcal{X}$. Verify whether each of the following statements is true or false.

(a) Let $E \subseteq \mathcal{X}$. Then, $TE \subseteq E$ if and only if $T^{-1}E \supseteq E$.

(b) For every $A, B, C \subseteq \mathcal{X}$, we have $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$.

(c) For every $A, B \subseteq \mathcal{X}$, we have $T^{-1}A \setminus T^{-1}B = T^{-1}(A \setminus B)$.

(d) For every $A, B, I, J \subseteq \mathcal{X}$, we have $I \cap J \subseteq (A \cap B) \cup (I \setminus A) \cup (J \setminus B)$.

Problem 2 (Semi-invariant functions are invariant). Let $(\mathcal{X}, \mathscr{F}, \mu, T)$ be a measure-preserving dynamical system. Let $f : \mathcal{X} \to \mathbb{R}$ be a measurable function and suppose that $f\big(T(x)\big) \leq f(x)$ for μ-almost every $x \in \mathcal{X}$. Prove that in fact $f\big(T(x)\big) = f(x)$ for μ-almost every $x \in \mathcal{X}$.

Hint: Examine a finite-state dynamical system first.

Problem 3 (Proof of Kac's recurrence theorem). Complete the proof of Kac's theorem discussed in class. Namely,

(a) Verify that the first return time τ_A^+ to a measurable set $A \subseteq \mathcal{X}$ is measurable.

(b) Verify that $\widetilde{A} \cap T^{-1}A^{\mathsf{c}} = T^{-1}(\widetilde{A} \setminus A) \uplus (A \setminus T^{-1}\widetilde{A})$, where $\widetilde{A} \coloneqq \{x \in \mathcal{X} : T^n(x) \in A \text{ for some } n \geq 0\}$.

(c) Fix the bug in the "proof", namely, remove the unjustified assumption that τ_A^+ is integrable over $\widetilde{A}$.

Hint: Let M be a large number and repeat the reasoning for $\int_{\widetilde{A}} (\tau_A^+ \wedge M)\,\mathrm{d}\mu$ instead of $\int_{\widetilde{A}} \tau_A^+\,\mathrm{d}\mu$. At the end, let $M \to \infty$. (Here, $a \wedge b$ stands for the minimum of a and b.)

Problem 4 (A variant of Poincaré's theorem). Let μ be a probability measure on a measurable space $\mathcal{X}$ and $T\colon \mathcal{X} \to \mathcal{X}$ a measurable map that preserves μ. Let $A \subseteq \mathcal{X}$ be a measurable set with $\mu(A) > 0$.

(a) Prove that there exist a k with $0 < k \leq \lfloor 1/\mu(A) \rfloor$ such that $\mu(A \cap T^{-k}A) > 0$.

Hint: Consider $A, T^{-1}A, T^{-2}A, \ldots$.

(b) Use the previous result to give an alternative proof of the 1st claim in the standard version of Poincaré's theorem.

Problem 5 (Finite dynamical systems). Consider the finite-state dynamical system $(\mathcal{X}, T)$ defined in Problem 6 of Assignment 1. Identify all subsets of $\mathcal{X}$ that are [strictly/forward/backward] invariant. Given a T-invariant probability measure μ on $\mathcal{X}$, also identify all subsets of $\mathcal{X}$ that are invariant modulo μ.

Recall: $\mathcal{X} \coloneqq \{0, 1, 2, 3, 4\}$, and $T : \mathcal{X} \to \mathcal{X}$ is given by $T(0) \coloneqq 1$, $T(1) \coloneqq 2$, $T(2) \coloneqq 3$, and $T(3) \coloneqq 1$, $T(4) \coloneqq 4$.

Problem 6 (Ergodicity: Rational rotation). Let $\mathcal{X} \coloneqq \mathbb{R}/\mathbb{Z}$. Let α be a rational, and let $R_\alpha(x) \coloneqq x + \alpha \pmod 1$ be the *rotation-by-α* map on $\mathcal{X}$. Prove that R_α is not ergodic with respect to the Lebesgue measure. Give an example of a probability measure on $\mathcal{X}$ with respect to which R_α is ergodic.

Problem 7 (Ergodicity: Irrational rotation). Let $\mathcal{X} \coloneqq \mathbb{R}/\mathbb{Z}$. Let α be an irrational, and let $R_\alpha(x) \coloneqq x + \alpha \pmod 1$ be the *rotation-by-α* map on $\mathcal{X}$. Let λ denote the Lebesgue measure on $\mathcal{X}$.

(a) Prove Kronecker's theorem: For every $x \in \mathcal{X}$ and every non-empty open interval $(a, b) \subseteq \mathcal{X}$, there exists an $n > 0$ such that $R_\alpha^n(x) \in (a, b)$.

Hint: Pigeonhole principle.

(b) Let $I, J \subseteq \mathcal{X}$ be intervals, and let $\ell \in \mathbb{R}$ be such that $0 < \ell < \min\{\lambda(I), \lambda(J)\}$. Prove that $\lambda(I \cap R_\alpha^{-n} J) \geq \ell$ for some $n > 0$.

(c) Prove that R_α is ergodic with respect to λ.

Hint: Show that for every $A, B \subseteq \mathcal{X}$ with $\lambda(A) > 0$ and $\lambda(B) > 0$, there exists an $n > 0$ such that $\lambda(A \cap R_\alpha^{-n} B) > 0$. To this end, approximate A and B (mod λ) with intervals.