American University of Beirut

Introduction to Ergodic Theory

MATH 307K (2024–2025 Spring)

Assignment 1

Problem 1 (Binary expansions)**.** Prove that the binary expansion of a real number $x \in [0,1)$ is eventually periodic if and only if x is rational.

Problem 2 (Rational rotation)**.** Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ denote the *rotation-by-*α transformation given by $R_\alpha(x) \coloneqq x + \alpha \pmod 1$. Suppose that $\alpha = \ell/m$ is a rational ($\ell \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$ are relatively prime). During the lecture, we saw that for every interval $(a,b) \subseteq \mathbb{R}/\mathbb{Z}$ and every point $x \in \mathbb{R}/\mathbb{Z}$, the asymptotic frequency

$$d_{(a,b)}(x) \coloneqq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{(a,b)}\big(R_\alpha^k(x)\big)$$

of time steps at which the orbit of x enters (a,b) exists. What are the possible values of $d_{(a,b)}(x)$ when we vary x?

Problem 3 (Invariant measures: Rotation)**.** Let R_α denote the *rotation-by-*α map as in the previous problem.

(a) Prove that irrespective of the value of $\alpha \in \mathbb{R}$, the map R_α preserves the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$.

(b) Can you find another probability measure on $\mathbb{R}/\mathbb{Z}$ that is preserved by R_α?

Hint: Consider the cases in which α is rational and irrational separately. You may use Weyl's theorem if needed.

Problem 4 (Invariant measures: Multiplication by 2)**.** Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ denote the *multiplication-by-*2 transformation given by $T(x) \coloneqq 2x \pmod 1$. Prove that T preserves the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$.

Problem 5 (Bernoulli shift)**.** Let Γ be a finite alphabet, and $\mathcal{X} \coloneqq \Gamma^{\mathbb{N}}$ the space of all infinite sequences $x = x_0x_1x_2\cdots$ of symbols from Γ. A *cylinder set* in $\mathcal{X}$ is a set of the form

$$[u] \coloneqq \{x \in \mathcal{X} : x_0x_1\cdots x_{\ell-1} = u_0u_1\cdots u_{\ell-1}\}$$

where $u = u_0u_1\cdots u_{\ell-1} \in \Gamma^\ell$ is a *word* of length $\ell \in \mathbb{N}$ on Γ. The *product σ-algebra* on $\mathcal{X}$ is the σ-algebra generated by the cylinder sets.

(a) Verify that, together with the empty set, the cylinder sets form a semi-algebra.

The *product topology* on $\mathcal{X}$ is the topology generated by the cylinder sets.

(b) Verify that, in the product topology, the cylinder sets are both open and closed.

(c) Verify that the product topology on $\mathcal{X}$ is compact. *Hint:* Tychonoff's theorem.

(d) Verify that the product σ-algebra on $\mathcal{X}$ coincides with the Borel σ-algebra corresponding to the product topology.

Let $p : \Gamma \to [0,1]$ be a probability distribution on Γ (i.e., a function satisfying $\sum_{a\in\Gamma} p(a) = 1$).

(e) Show that there exists a unique probability measure π_p on $\mathcal{X}$ such that

$$\pi_p([u]) = \prod_{k=0}^{\ell-1} p(u_k)$$

for every word $u = u_0u_1\cdots u_{\ell-1} \in \Gamma^\ell$ with $\ell \in \mathbb{N}$. This is called the *Bernoulli* measure (a.k.a. the *product* measure) with marginal p.

Hint: Verify that π_p defined as above is finitely additive on the semi-algebra of cylinder sets and argue that countable additivity follows trivially.

Let $\sigma : \mathcal{X} \to \mathcal{X}$ be the *shift* map defined by $\sigma(x)_k \coloneqq x_{k+1}$.

(f) Verify that the shift map preserves every Bernoulli measure on $\mathcal{X}$.

Problem 6 (Finite dynamical systems)**.** Let $\mathcal{X} := \{0, 1, 2, 3, 4\}$ and consider the map $T : \mathcal{X} \to \mathcal{X}$, where $T(0) := 1$, $T(1) := 2$, $T(2) := 3$, and $T(3) := 1$, $T(4) := 4$.

(a) Identify all the probability measures on $\mathcal{X}$ that are preserved by T.

(b) Prove that for every function $f : \mathcal{X} \to \mathbb{R}$ and every point $x \in \mathcal{X}$, the limit

$$\overline{f}(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\big(T^k(x)\big)$$

exists, and find its value.

Problem 7 (Proof of Poincaré's recurrence theorem)**.** Prove the second claim in Poincaré's theorem. Namely, let μ be a probability measure on a measurable space $\mathcal{X}$ and $T\colon \mathcal{X} \to \mathcal{X}$ a measurable map that preserves μ. Let $A \subseteq \mathcal{X}$ be a measurable set with $\mu(A) > 0$. Prove that the orbit of μ-a.e. $x \in A$ returns to A *infinitely many times*.

Hint: Either adapt the proof of the 1st claim, or use the 1st claim as a lemma.