General information

Introduction to Ergodic Theory (MATH 307K)

Syllabus:

- math307K-syllabus

Time and location:

- MWF 11:00am–11:50am
- Bliss Hall 206

Instructor: Siamak Taati

- Office: Bliss Hall 312B
- Email: st71@aub.edu.lb

Office hours:

- Thursdays 1:00pm--3:00pm

Notes

Kac's recurrence theorem

Von Neumann's ergodic theorem

Birkhoff's ergodic theorem

Last update: March 7, 2022

Mixing

(partial)

Last update: March 10, 2022

Assignments

Assignment 1

To be discussed on Friday, February 4

Note [2022-01-30]: For Problem 3(b), you may use Weyl's theorem if needed. There was a typo in Problem 6(b) which is now fixed (namely, it should be $f : \mathcal{X} \to \mathbb{R}$ instead of $f : \mathcal{X} \to \mathcal{X}$).

Assignment 2

To be discussed on ~~Friday, February 25~~ Monday, February 28

Assignment 3

To be discussed on Friday, March 18

Note [2022-03-09]: A new question is added.

Projects

Guidelines

Some suggested topics for presentation/final paper

Last update: March 17, 2022

Polls

Time of extra session (Weeks 6-7)

Please respond by Tuesday, March 1, evening.

Time of extra session (Week 12)

Please respond by Monday, April 11, noon.

Poll results (week 12)

Time of extra session (Week 13)

Please respond by Wednesday, April 13, noon.

Poll results (week 13)

Doodle poll for the presentation sessions

Please respond by Sunday, May 1, evening.

Final presentations

Schedule [updated: 2022-05-06]:

- Roa and Jana: Monday 10:15-11:00
- Estepan: Monday 11:00-11:45
- Zeinab and Lara: Monday 11:45-12:30
- Youmna: Tuesday 13:00-13:30
- Hiba and Sara: Tuesday 13:30-14:15
- Ali: Tuesday 14:15-14:45
- Hadi: Tuesday 14:45-15:15

Location [updated: 2022-05-05]:

- Monday: Bliss 206 (booked)
- Tuesday: Bliss 206 (if available; if not, check my office)

Week 14

Note [2022-04-29]: A poll for the schedule of the presentation sessions is posted (see the Polls section above). Please respond by Sunday evening.

Friday 2022-04-29

- Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.
  - The entropy of a discrete random variable $X : \Omega \to \Gamma$ is

    $$H(X) := -\sum_{a\in\Gamma} \mathbb{P}(X=a)\log\mathbb{P}(X=a)\ ,$$

    with the convention $0\log 0 = 0$.
  - The conditional entropy of discrete RV $X : \Omega \to \Gamma$ given a discrete RV $Y : \Omega \to \Lambda$ is

    $$\begin{aligned} H(X\,|\,Y) &:= -\sum_{b\in\Lambda}\sum_{a\in\Gamma} \mathbb{P}(X=a, Y=b)\log\mathbb{P}(X=a) \\ &= \sum_{b\in\Lambda} H(X\,|\,Y=b)\ . \end{aligned}$$

- Basic properties of entropy and conditional entropy
- The conditional entropy $H(X\,|\,\mathscr{A})$ of a RV X given a sub-σ-algebra $\mathscr{A} \subseteq \mathscr{F}$ is

  $$H(X\,|\,\mathscr{A}) := -\int \sum_{a\in\Gamma} \mathbb{P}(X=a\,|\,\mathscr{A})\log\mathbb{P}(X=a\,|\,\mathscr{A})\,\mathrm{d}\mathbb{P}\ .$$

- An information source can be thought of as a process

  $$\ldots, W_{-2}, W_{-1}, W_0, W_1, W_2, \ldots$$

  with values in a finite set Γ.
- The *entropy rate* $h\left((W_n)_{n\in\mathbb{Z}}\right)$ of a process $(W_n)_{n\in\mathbb{Z}}$ can be defined as

  $$\lim_{n\to\infty}\frac{1}{2n+1}H(W_{-n}, W_{-n+1}, \ldots, W_n)\ ,$$

  or as

  $$H\left(W_m\,|\,\ldots, W_{m-2}, W_{m-1}\right)\ .$$

- Theorem: For a stationary process, the above two definitions are (well-defined and) equivalent.
- The entropy of a measure-preserving dynamical system is defined in terms of the processes obtained by partially observing the system.
- Definition (Kolmogorov-Sinai entropy): Let $(\mathcal{X}, \mathscr{F}, \mu, T)$ be an invertible measure-preserving dynamical system. Let $\varphi : \mathcal{X} \to \Gamma$ be a *finite-valued observable* (i.e., a measurable map into a finite set Γ). The entropy rate of $(\mathcal{X}, \mathscr{F}, \mu, T)$ with respect to φ is

  $$h_\mu(\mathcal{X}, T; \varphi) := h\left((W_n)_{n\in\mathbb{Z}}\right)$$

  where $W_n := \varphi\circ T^n$. The *Kolmogorov-Sinai entropy* of $(\mathcal{X}, \mathscr{F}, \mu, T)$ is

  $$h_\mu(\mathcal{X}, T) := \sup\left\{h_\mu(\mathcal{X}, T; \varphi) : \varphi \text{ a finite-valued observable}\right\}\ .$$

- Isomorphism between measure-preserving dynamical systems
- Remark: The KS-entropy is invariant under isomorphisms.
- Exercise 1: Show that the KS-entropy of a Bernoulli shift $(\Sigma^{\mathbb{Z}}, \mathscr{F}, \mu_p, \sigma)$ is $H(p)$.
- Exercise 2: Show that the KS-entropy of every (rational or irrational) rotation is 0.

Wednesday 2022-04-27

- Completion of the proof of the ergodic decomposition theorem (when f is not continuous).
- Proposition: The unique invariant measure of every uniquely ergodic system is ergodic.
- Proposition: In a uniquely ergodic system, every point is generic for the unique invariant measure.
- Remark: Let $(\mathcal{X}, T)$ be a uniquely ergodic system with unique invariant measure π. If $f : \mathcal{X} \to \mathbb{R}$ is not continuous, then $\frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x))$ need not converge to $\pi(f)$ for every x, even though it does for π-a.e. x.
- Theorem (Characterizations of unique ergodicity): Let $(\mathcal{X}, T)$ be a topological system. The following are equivalent:
  a. $(\mathcal{X}, T)$ is uniquely ergodic.
  b. There exists $\pi \in \mathscr{P}(\mathcal{X}, T)$ such that every point $x \in \mathcal{X}$ is generic for π.
  c. For every $f \in C(\mathcal{X})$, the sequence $\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$ converges pointwise to a constant.
  d. For every $f \in C(\mathcal{X})$, the sequence $\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$ converges uniformly to a constant.
- The concepts of
  - The *entropy* $H(X)$ of a discrete RV X (a measure of "uncertainty" about the value of X).
  - The *conditional entropy* $H(X \mid Y) = H(X, Y) - H(Y)$ of a discrete RV X given another discrete RV Y (on average, how much "uncertainty" is left about X if we learn the value of Y).
  - The *mutual information* $I(X; Y) = H(X) - H(X \mid Y)$ between X and Y (on average, how much information does Y have about X, or vice versa).
- An axiomatic definition of entropy based on its desired properties leads to the expression

$$H(X) = -\sum_{a \in \Gamma} \mathbb{P}(X = a) \log \mathbb{P}(X = a)$$

  where Γ denotes the (finite/countable) set of possible values of X.
- Theorem (an interpretation of entropy): Let L denote the smallest average number of yes-no questions needed to determine X. Then, $H(X) \leq L \leq H(X) + 1$. (Proof can be found in any information theory textbook.)
- In particular, if $X_1, X_2, \ldots, X_n$ are independent copies of X and L_n denotes the smallest average number of yes-no questions needed to determine $X_1, X_2, \ldots, X_n$, then $\lim_{n \to \infty} L_n / n = H(X)$.

Monday 2022-04-25

- Holiday; no class

Week 13

Friday 2022-04-22

- Holiday; no class

Wednesday 2022-04-20

- Proof of the theorem from last time: In a topological system $(\mathcal{X}, T)$, the set $E(\mathcal{X}, T)$ of points that are generic for some ergodic measure is measurable and has measure one with respect to every invariant measure.
- Proof of the ergodic decomposition theorem (topological variant).
- Example (uniquely ergodic and minimal):
  - Every irrational rotation is uniquely ergodic with the Lebesgue measure as the unique invariant measure.
  - The dyadic adding machine is uniquely ergodic with the uniform Bernoulli measure as the unique invariant measure.
- Example (uniquely ergodic but not minimal):
  - Recall: If $(\mathcal{X}, T)$ is uniquely ergodic and its invariant measure has full support, then $(\mathcal{X}, T)$ is also minimal.
  - The finite system with space $\mathcal{X} := \{0, 1\}$ and transformation $(0 \mapsto 1; 1 \mapsto 1)$ is uniquely ergodic but not minimal.
- There are also systems that are minimal but not uniquely ergodic. (Stay tuned for the presentation of Zeinab and Lara.)

Tuesday 2022-04-19 (extra session)

- In the analogy between measure-preserving and topological settings, generic points to ergodic systems are like transitive points to transitive systems.
- Exercise: Let $(\mathcal{X}, T)$ be a topological system and suppose $\mu \in \mathscr{P}(\mathcal{X}, T)$ is a full-support ergodic measure. We have

already seen that the system is necessarily transitive. Prove that every generic point for μ is transitive. In particular, the set of transitive points has measure 1.

- Generic points in shift dynamical systems: Let $(\Sigma^{\mathbb{N}}, \sigma)$ be a full shift. A configuration x is generic for a shift-invariant measure μ if and only if for every finite word $u \in \Sigma^*$, the asymptotic frequency of the occurrences of u in x is $\mu([u])$.
- Example 1 (An explicit example of a generic point): Let $\mu_{1/2}$ be the uniform Bernoulli measure on $\{0, 1\}^{\mathbb{N}}$. The following configuration is $\mu_{1/2}$-generic:

$$x := 0\,1\,00\,01\,10\,11\,000\,001\,\cdots$$

- Example 2 (A measure with a null set of generic points): Let $\mu := \lambda\mu_{1/3} + (1-\lambda)\mu_{2/3}$ where $\mu_{1/3}, \mu_{2/3}$ are Bernoulli measures on $\{0, 1\}^{\mathbb{N}}$ with parameters $1/3, 2/3$, and $0 < \lambda < 1$. Then, μ-almost every point is generic for either $\mu_{1/3}$ or $\mu_{2/3}$. In particular, the set of generic points for μ is a μ-null set. However, this set is still non-empty.
- Example 3 (An example of a point which is generic for a non-ergodic measure): Let $\mu := \frac{1}{2}\delta_{\underline{0}} + \frac{1}{2}\delta_{\underline{1}}$ where $\underline{0} := 0000\cdots$ and $\underline{1} := 1111\cdots$. he following configuration is μ-generic:

$$x := 0\,1\,0000\,1111\,000000000\,111111111\,\cdots$$

- Example 4 (A measure with no generic point): Consider a finite system with two distinct cycles (e.g., the one in Assignment 1). Let μ_1 and μ_2 be two ergodic measure supported at two distinct cycles. Then, any invariant measure which is a non-trivial convex combination of μ_1 and μ_2 has no generic point.
- Proposition: Let $\mathcal{X}, T$ be a topological system and $\mu \in \mathscr{P}(\mathcal{X}, T)$. Then, μ-almost every point is generic for some measure in $\mathscr{P}(\mathcal{X}, T)$. (Proof sketched.)
- Theorem: Let $\mathcal{X}, T$ be a topological system and $\mu \in \mathscr{P}(\mathcal{X}, T)$. Then, μ-almost every point is generic for some *ergodic* measure in $\mathscr{P}(\mathcal{X}, T)$. (Proof to come.)
- The recording (password: [redacted]) and the board of the session

Monday 2022-04-18

- Holiday; no class

Week 12

Note [2022-04-15]: Following the poll, the extra session will be on Tuesday, April 19, 15:00-16:00 (online).

Friday 2022-04-15

- Holiday; no class

Note [2022-04-13]: Please note that due to personal engagements, I will not be able to have office hours this Thursday. Feel free to contact me by email, or drop by on a different day.

Wednesday 2022-04-13

- Example: In a finite dynamical system (e.g., the one in Problem 6 of Assignment 1), there is an ergodic measure corresponding to each cycle in the transition graph, and each invariant measure can be written as a convex combination of such measures.
- Ergodic decomposition theorem (topological variant): Let $\mathcal{X}$ be a compact metric space and $T : \mathcal{X} \to \mathcal{X}$ a continuous map. There exists a measurable set $\mathcal{X}_0$ and a map $x \in \mathcal{X}_0 \mapsto \nu_x \in \mathscr{P}(\mathcal{X}, T)$ such that
    i. $\mu(\mathcal{X}_0) = 1$ for every $\mu \in \mathscr{P}(\mathcal{X}, T)$,
    ii. For every $x \in \mathcal{X}_0$, the measure ν_x is ergodic,
    iii. $T^{-1}\mathcal{X}_0 = \mathcal{X}_0$ and $\nu_{T(x)} = \nu_x$ for each $x \in \mathcal{X}_0$,
    iv. For every bounded measurable $f : \mathcal{X} \to \mathbb{R}$,
        a. The map $x \mapsto \nu_x(f) = \int f\,\mathrm{d}\nu_x$ is measurable,
        b. For every $\mu \in \mathscr{P}(\mathcal{X}, T)$,

$$\int f\,\mathrm{d}\mu = \int_{\mathcal{X}_0} \left(\int f\,\mathrm{d}\nu_x\right) \mathrm{d}\mu(x)$$

- Remark: The above theorem says that every invariant measure μ can be written as a convex mixture of ergodic measures

$$\mu(\cdot) = \int_{\mathcal{X}_0} \nu_x(\cdot)\, \mathrm{d}\mu(x)\ .$$

  Compare this with
  1. The earlier theorem (based on Krein-Milman theorem) which said that every invariant measure is a limit of convex linear combinations of ergodic measures,
  2. The earlier observation that every (non-invariant) measure is a convex mixture of Dirac measures.
- Generic points: Let $\mathcal{X}, T$ be a topological system. We say that a point $x \in \mathcal{X}$ is *generic* for a measure $\mu \in \mathscr{P}(\mathcal{X}, T)$ if for every $f \in C(\mathcal{X})$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x)) \to \int f\,\mathrm{d}\mu \quad \text{as } n \to \infty\ .$$

  Equivalently, x is ergodic for μ if

$$\frac{1}{n}\sum_{k=0}^{n-1} T^k\delta_x \to \mu \quad \text{in weak* as } n \to \infty\ .$$

  We denote by G_μ the set of all points that are generic for μ.
- Proposition: Let $\mathcal{X}, T$ be a topological system and $\mu \in \mathscr{P}(\mathcal{X}, T)$ be an ergodic measure. Then, μ-a.e. point is generic for μ. More specifically, G_μ is measurable and $\mu(G_\mu) = 1$.
- Theorem (Characterizations of ergodicity; topological setting): Let $\mathcal{X}, T$ be a topological system and $\mu \in \mathscr{P}(\mathcal{X}, T)$. The following are equivalent:
  i. μ is ergodic.
  ii. For μ-a.e. $x \in \mathcal{X}$, we have $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{T^k(x)} \to \mu$.
  iii. For μ-a.e. $x \in \mathcal{X}$ and every $f \in C(\mathcal{X})$, we have $\frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x)) \to \mu(f)$.
  iv. For every $f \in C(\mathcal{X})$ and μ-a.e. $x \in \mathcal{X}$, we have $\frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x)) \to \mu(f)$.

Tuesday 2022-04-12 (extra session)

- Proposition: If $\mathcal{X}$ is a compact metric space and $T : \mathcal{X} \to \mathcal{X}$ is a continuous map, then the set $\mathscr{P}(\mathcal{X}, T)$ of T-invariant Borel probability measures on $\mathcal{X}$ is non-empty and compact.
- Theorem (geometry of $\mathscr{P}(\mathcal{X}, T)$): Let $\mathcal{X}$ be a measurable space and $T : \mathcal{X} \to \mathcal{X}$ a measurable map. Then, (a) $\mathscr{P}(\mathcal{X}, T)$ is convex, (b) $\mu \in \mathscr{P}(\mathcal{X}, T)$ is ergodic if and only if it is extremal, (c) Distinct ergodic measures $\mu, \nu \in \mathscr{P}(\mathcal{X}, T)$ are mutually singular.
- Theorem: Let $\mathcal{X}$ be a compact metric space and $T : \mathcal{X} \to \mathcal{X}$ a continuous map. Then, $\mathscr{P}(\mathcal{X}, T)$ is the closed convex hull of the set $\mathscr{P}_{\mathrm{e}}(\mathcal{X}, T)$ of ergodic measures. (This is a corollary of the Krein-Milman theorem.)
- The recording (password: [redacted]) and the board of the session

Note [2022-04-11]: Following the poll, the extra session will be on Tuesday, April 12, 14:00-15:00 (online).

Monday 2022-04-11

- Portmanteau theorem (characterizations of weak* convergence): Remark (3) on page 149 of Walters, or Wikipedia
- Theorem: Let $\mathcal{X}$ be a compact metric space. Then, $\mathscr{P}(\mathcal{X})$ is also compact and metrizable.
- Remarks: Let $\mathcal{X}$ be a compact metric space. Then, (a) $x \mapsto \delta_x$ is an embedding of $\mathcal{X}$ into $\mathscr{P}(\mathcal{X})$, (b) $\mathscr{P}(\mathcal{X})$ is convex, (c) The Dirac measures are precisely the extremal elements of $\mathscr{P}(\mathcal{X})$, (d) Every $\mu \in \mathscr{P}(\mathcal{X})$ can be approximated by convex combinations of Dirac measures, (e) Every $\mu \in \mathscr{P}(\mathcal{X})$ is a convex mixture of Dirac measures, in the sense that, for every bounded measurable $f : \mathcal{X} \to \mathbb{R}$, we have

$$\int f\,\mathrm{d}\mu = \int \left(\int f\,\mathrm{d}\delta_x\right) \mathrm{d}\mu(x)\ .$$

- Exercise: Prove the missing arguments for the latter remarks.
- Proposition: Let $\mathcal{X}$ be a compact metric space, $T : \mathcal{X} \to \mathcal{X}$ a continuous map, and $\mu \in \mathscr{P}(\mathcal{X})$ a probability measure.

Then, $T\mu = \mu$ if and only if $\mu(f \circ T) = \mu(f)$ for every $f \in C(\mathcal{X})$.

- Proposition: Let $\mathcal{X}$ be a compact metric space and $T : \mathcal{X} \to \mathcal{X}$ a continuous map. The map $\mu \mapsto T\mu$ on $\mathscr{P}(\mathcal{X})$ is continuous and affine.

Week 11

Note [2022-04-10]: Two polls for the schedule of the extra sessions in Weeks 12 and 13 are posted (see the Polls section above). Please respond the first by Monday noon and the second by Wednesday noon.

Friday 2022-04-08

- Notation: We denote by $\mathscr{P}(\mathcal{X})$ the set of Borel probability measures on a topological space $\mathcal{X}$. We write $C(\mathcal{X})$ the Banach space of bounded continuous functions $f : \mathcal{X} \to \mathbb{R}$ on a topological space $\mathcal{X}$ (with the uniform norm). *Recall*: if $\mathcal{X}$ is compact, then all continuous functions are bounded.
- Proposition: Let $\mathcal{X}$ be a metric space and $\mu, \nu \in \mathscr{P}(\mathcal{X})$ with $\mu \neq \nu$. Then, there exists an $f \in C(\mathcal{X})$ such that $\mu(f) \neq \nu(f)$. Exercise: Prove this.
- A linear function $J : C(\mathcal{X}) \to \mathbb{R}$ is (1) *positive* if $J(f) \geq 0$ whenever $f \geq 0$ and (2) *normalized* if $J(1_{\mathcal{X}}) = 1$.
- Proposition: Let $\mathcal{X}$ be a metric space. Every positive linear functional $J : C(\mathcal{X}) \to \mathbb{R}$ is continuous.
- Riesz representaiton theorem: Let $\mathcal{X}$ be a compact metric space. For every normalized positive linear functional $J : C(\mathcal{X}) \to \mathbb{R}$, there exists a (unique) Borel probability measure μ such that $\mu(f) = J(f)$ for every $f \in C(\mathcal{X})$.
- The *weak* topology* on $\mathscr{P}(\mathcal{X})$ is the smallest topology that makes all the maps $\mu \mapsto \mu(f)$ (for $f \in C(\mathcal{X})$) continuous.
- Convergence in weak* topology: $\mu_n \to \mu$ if and only if $\mu_n(f) \to \mu(f)$ for every $f \in C(\mathcal{X})$.
- Proposition: Let $\mathcal{X}$ be a compact metric space. Then, the weak* topology on $\mathscr{P}(\mathcal{X})$ is metrizable.
- Remark (follows from the Stone-Weierstrass theorem): If $\mathcal{X}$ is a compact metric space, then $C(\mathcal{X})$ contains a countable dense subset.

Wednesday 2022-04-06

- Proposition: Every topological system has a minimal subsystem (proof via Zorn's lemma).
- Birkhoff's recurrence theorem: Every topological system has at least one almost periodic point.
- Poincaré's recurrence theorem (topological version): Let $\mathcal{X}$ be a compact metric space, $T : \mathcal{X} \to \mathcal{X}$ a continuous map, and μ a Borel probability measure on $\mathcal{X}$ that is invariant under T. Then, μ-a.e. point in $\mathcal{X}$ is (topologically) recurrent.
- Proposition (proof comes later): Every topological system admits at least one invariant probability measure.
- *Topological support* of a positive Borel measure μ on a topological space $\mathcal{X}$ is

$$\mathrm{supp}(\mu) := \{x \in \mathcal{X} : \mu(B) > 0 \text{ for every open } B \ni x\} .$$

  We say that μ has full support if $\mathrm{supp}(\mu) = \mathcal{X}$.
- Exercise: If $\mathcal{X}$ is a compact metric space and μ a Borel probability measure on $\mathcal{X}$, then $\mu(\mathrm{supp}(\mu)) = 1$.
- Proposition: Let $(\mathcal{X}, T)$ be a topological system and μ a T-invariant Borel probability measure on $\mathcal{X}$ which has full support. (1) If μ is ergodic, then T is transitive. (2) If μ is the only T-invariant Borel probability measure on $\mathcal{X}$, then T is minimal.
- A topological system is called *uniquely ergodic* if it admits exactly one T-invariant Borel probability measure.

Monday 2022-04-04

- Updates on the projects:
- **Estepan**: An example of a weakly mixing system which is not strongly mixing.
- **Hiba** and **Sara**: An ergodic-theoretic proof of van der Waerden's theorem.
- **Jana** and **Roa**: Spectral methods and applications to isomorphism problem.
- **Ali**: Fourier analytic proof of Weyl's theorem and similar results on compact Abelian groups.
- **Hadi**: The randomization property of the XOR cellular automaton.
- **Youmna**: An alternative proof of the pointwise ergodic theorem.
- **Zeinab** and **Lara**: An example of a minimal system which is not uniquely ergodic.
- Theorem (Birkhoff): In every minimal system, every point is almost periodic.

Week 10

Friday 2022-04-01

- Minimal systems
- The *orbit closure* $E := \overline{\mathcal{O}^+(z)}$ of a point z in a topological system $(\mathcal{X}, T)$ is non-empty, closed and forward-invariant, hence defines a subsystem (E, T).
- Theorem (characterizations of minimality): Let $(\mathcal{X}, T)$ be a topological system. The following are equivalent: (a) T is minimal. (b) Every point in $\mathcal{X}$ is a transitive point for T. (c) For every non-empty open set $A \subseteq \mathcal{X}$, we have $\bigcup_{n \geq 1} T^{-n} A = \mathcal{X}$.
- Exercise: Prove the latter theorem.
- Example: Irrational rotations are minimal. (This is Kronecker's theorem.)
- Example: A vertex shift is minimal if and only if its underlying graph consists of a single cycle.
- Example: The XOR cellular automaton is not minimal because $x = \cdots 000 \cdots$ is a fixed point.
- Example: The dyadic adding machine is minimal.
- Recurrent points, almost periodic points and quasi-periodic points.
- Example: Every point in the dyadic adding machine is quasi-periodic. Every point in an irrational rotation is almost-periodic (but not quasi-periodic).

Wednesday 2022-03-30

- Theorem (characterizations of transitivity): Let $(\mathcal{X}, T)$ be a topological system. The following are equivalent: (a) T is transitive. (b) Every closed, forward invariant subset of $\mathcal{X}$ is either nowhere dense or the entire $\mathcal{X}$ itself. (c) T has a transitive point. (d) T has a residual set of transitive points.

Note [2022-03-28]: The Monday, April 4 session is dedicated to discussing the progress in your projects. We will have a couple of extra sessions in the coming weeks.

Monday 2022-03-28

- Topological transitivity and topological mixing
- Example: The golden-mean shift is topologically mixing.
- Proposition: (1) A vertex/edge shift is topologically transitive if and only if the underlying graph is strongly connected. (2) A vertex/edge shift is topologically mixing if and only if the underlying graph is strongly connected and aperiodic.
- Exercise: The XOR cellular automaton is topologically mixing. Hint: use the fact that $T(x \oplus y) = T(x) \oplus T(y)$, where $\oplus$ stands for site-wise addition modulo 2.
- Example: The dyadic adding machine is topologically transitive but not mixing.
- A point x in a topological dynamical system $(\mathcal{X}, T)$ is called a (forward) *transitive point* if for every non-empty open set $A \subseteq \mathcal{X}$, there exists an $n \geq 1$ such that $T^n(x) \in A$.
- Example: A transitive point in the full shift.
- Exercise: Find a transitive point in the golden-mean shift.
- Proposition: The trajectory of every transitive point visits every non-empty open set infinitely many times.

Week 9

Monday 2022-03-25

- Holiday; no class

Wednesday 2022-03-23

- Exercise session

Monday 2022-03-21

- Exercise session

Week 8

Friday 2022-03-18

- A subsystem of a full shift is called a *subshift* (or a *shift*), and its underlying space a *shift space*.
- Example: Golden-mean shift
- Example: Vertex and edge shifts associated to a finite directed graph $G = (V, E)$
- Example: The subshift defined by forbidding the words in a set $F \subseteq \Sigma^*$, that is, $(\mathcal{X}_F, \sigma)$ where

  $$\mathcal{X}_F := \{x \in \Sigma^{\mathbb{Z}} : x_k x_{k+1} \cdots x_{k+\ell-1} \notin F \text{ for all } k, \ell\}$$

  (provided the latter set is non-empty).
- Exercise: Every sbushift is of the above form. (Note: the choice of F is not unique.)
- If F is finite, then X_F is called a *subshift of finite type* (*SFT*).
- Example: The XOR cellular automaton is the system $\{0, 1\}^{\mathbb{Z}}, T$ where $T(x)_k := x_k + x_{k+1} \pmod 2$. Exercise: Show that (1) T is continuous, (2) $T \circ \sigma = \sigma \circ T$, and (3) T is onto but not one-to-one.
- Example: The dyadic adding machine is the system $\{0, 1\}^{\mathbb{Z}}, T$ where

  $$T(x)_k := \begin{cases} x_k + 1 \pmod 2 & \text{if } x_{k-1} = x_{k-2} = \cdots = x_0 = 1 \\ x_k & \text{otherwise.} \end{cases}$$

  Exercise: Verify that T is continuous.
- A topological system $(\mathcal{X}, T)$ is *(topologically) transitive* if for every pair of non-empty open sets $A, B \subseteq \mathcal{X}$, there exists an $n \geq 0$ such that $A \cap T^{-n}B \neq \varnothing$. Exercise: Verify that for $k \in \mathbb{N}$ modifying the condition $n \geq 0$ into $n \geq k$ gives an equivalent definition.
- Example: Irrational rotations are transitive; rational rotations are not.
- Example: Every full shift is transitive.
- Example: The golden-mean shift is transitive.

Wednesday 2022-03-16

- Interpretation of a Markov shift associated to a stochastic matrix Q and a probability distribution p satisfying $pQ = p$
- Theorem 1: If Q is irreducible, then (1) Q has a unique stationary distribution p and (2) the Markov shift associated to Q and p is ergodic.
- Theorem 2: Suppose Q is irreducible. The following are equivalent: (1) Q is aperiodic. (2) The Markov shift defined by Q is weakly mixing. (3) The Markov shift defined by Q is strongly mixing.
- Introduction to topological dynamics
- Example: one-sided full shift $\Sigma^{\mathbb{N}}, \sigma$ and two-sided full shift $\Sigma^{\mathbb{Z}}, \sigma$
- Exercise: Verify that the shift map is continuous.
- Subsystems of a topological dynamical system

Monday 2022-03-14

- Proof of "if $T \times T$ is ergodic, then T is weakly mixing."
- Theorem: T is strongly mixing if and only if for every measurable A, we have $\mu(A \cap T^{-n}A) \to \mu(A)^2$ as $n \to \infty$. Similar statements hold for weak mixing and ergodicity.
- Example: Markov measures

Week 7

Friday 2022-03-11

- Proof of Theorem 1.20 of Walters via the full-density diagonal lemma
- Theorem: Characterization of weak mixing in terms of convergence in density
- Proposition: The product of any two weakly mixing systems is again weakly mixing.
- Theorem: A measure-preserving map T is weakly mixing if and only if $T \times T$ is ergodic.

Note [2022-03-10]: Partial notes on mixing properties are posted (see the Notes section above).

Note [2022-03-10]: Partial notes on mixing properties are posted (see the Notes section above).

Note [2022-03-09]: The 3rd assignment is updated with a new question (see the Assignments section above).

Wednesday 2022-03-09

- Density of sets $J \subseteq \mathbb{N}$ in $\mathbb{N}$
- Lemma (full density diagonal sequence): Given a sequence $\mathbb{N} = J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$ of full density sets, there exist a full density set $J \subseteq \mathbb{N}$ which is "eventually a subset" of each J_n (i.e., $J \setminus J_n$ is finite for each n).
- An application of the lemma: Theorem 1.20 of Walters

Tuesday 2022-03-08 (extra session)

- "Mixing" characterization of ergodicity
- Weak and strong mixing
- Exercise: Irrational rotations are not weakly mixing.
- Every Bernoulli shift is strongly mixing (argument via approximation).
- The recording (password: [redacted]) and the board of the session

Note [2022-03-08]: The 3rd assignment is posted (see the Assignments section above).

Monday 2022-03-07

- Proof of the L^1 decomposition lemma
- Characterizations of I_1 and $\overline{B}_1$

Week 6

Friday 2022-03-04

- Notion of conditional expectation and its basic properties
- Exercise: Let $(\mathcal{X}, \mathscr{F}, \mu, T)$ be a measure-preserving system. Verify that

$$\mathscr{I}_\mu := \{E \in \mathscr{F} : \mu(E \Delta T^{-1} E) = 0\}$$

  is a σ-algebra.

Note [2022-03-03]: Notes on Birkhoff's ergodic theorem are updated so as to include the decomposition lemma (see the Notes section above).

Note [2022-02-28]: Following the poll, the extra session will be on Tuesday, March 8, 14:00-15:00 (online).

Wednesday 2022-03-02

- Exercise session

Note [2022-02-28]: A poll for the schedule of the extra session is posted (see the Polls section above). Please respond by Tuesday evening.

Monday 2022-02-28

- Exercise session
- Problem 3(b) onward are left for Wednesday.

Week 5

Note [2022-02-27]: Guidelines and suggested topics for the presentation/final paper are posted (see the Projects section above).

Note [2022-02-26]: Notes on Birkhoff's ergodic theorem are posted (see the Notes section above).

- The proof of the decomposition lemma is to be added.
- The proof of the maximal ergodic inequality (variant II) as presented in class had a bug, which is fixed in the notes.

Friday 2022-02-25

- Proof of the maximal ergodic inequality (two variants)

Wednesday 2022-02-23

- Proof of the L^2 decomposition lemma
- Statement of the L^1 decomposition lemma
- Statement of the maximal ergodic inequality
- Proof of Birkhoff's theorem using the decomposition lemma and the maximal ergodic inequality

Monday 2022-02-21

- Characterizations of ergodicity: Proof of the remaining implication
- Proof of von Neumann's ergodic theorem (except the decomposition lemma)
- Notes on von Neumann's theorem are posted (see the Notes section above).

Week 4

Friday 2022-02-18

- Characterizations of ergodicity (Theorems 1 and 2 from Wednesday): Proof of all implications except one

Wednesday 2022-02-16

- When is $\overline{f}(x) := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x))$ almost everywhere constant? $\rightsquigarrow$ case of finite dynamical systems.
- *Ergodicity*: A measure-preserving DS $(\mathcal{X}, \mathscr{F}, \mu, T)$ is ergodic if every strictly invariant set $E \subseteq \mathcal{X}$ is trivial (i.e., $\mu(E) = 0$ or $\mu(E) = 1$).
- Terminology for measurable sets: strictly T-invariant, forward T-invariant, backward T-invariant, T-invariant modulo μ
- Exercise: Identify the [strictly/forward/backward] invariant sets and invariant sets mod μ in the finite system of Problem 6 from Assignment 1.
- Terminology for measurable functions: T-invariant and T-invariant modulo μ
- Theorem 1: Five extra characterizations of ergodicity in terms of measurable sets (Theorem 1.5 of Walter's book).
- Theorem 2: Two extra characterizations of ergodicity in terms of measurable functions $f : \mathcal{X} \to \mathbb{R}$ (Theorem 1.6 of Walter's book).
- Corollary of Birkhoff's ergodic theorem for ergodic systems.

Note [2022-02-15]: The 2nd assignment is posted (see the Assignments section above).

Monday 2022-02-14

- Holiday; no class

Week 3

Friday 2022-02-11

- Exercise session

Wednesday 2022-02-09

- Holiday; no class

Monday 2022-02-07

- Exercise session
- Problems 5 and 6(partial) are left for Friday.

Week 2

Friday 2022-02-04

- Birkhoff's (pointwise) ergodic theorem
- Von Neumann's L^2 ergodic theorem
- L^p ergodic theorem
- The asymptotic average is invariant over each orbit.
- Proof of L^p ergodic theorem using Birkhoff's theorem (via approximation)

Note [2022-02-02]: The 1st exercise session is moved from Friday to Monday, February 7. On Friday, we will have a normal lecture.

Wednesday 2022-02-02

- Is Poincaré's theorem consistent with the Second Law of Thermodynamics?
- Kac's recurrence theorem and its "proof"
- The recording (password: [redacted]) and the board of the session
- Notes on Kac's recurrence theorem are posted (see the Notes section above). (They include the exercises mentioned during the lecture.)

Monday 2022-01-31

- More examples of measure-preserving dynamical systems: Bernoulli shifts, stationary processes, rotations on a compact group.
- Poincaré's Recurrence Theorem: Let $(\mathcal{X}, \mathscr{F}, \mu, T)$ be a measure-preserving dynamical system. Let $A \subseteq \mathcal{X}$ be a measure set with $\mu(A) > 0$. Then, for μ-a.e. $x \in A$, there exists a time $n > 0$ such that $T^n(x) \in A$. In fact, the orbit of μ-a.e. $x \in A$ returns to A infinitely many times. (*Note:* Recall that μ is a probability measure.)
- Exercise: Prove the 2nd claim in Poincaré's Recurrence Theorem (either by adapting the proof of the 1st part, or using the 1st part as a lemma).

Week 1

Note [2022-01-29]: The 1st assignment is posted (see the Assignments section above).

Friday 2022-01-28

- A *measure-preserving dynamical system* is described by (1) a measurable space $(\mathcal{X}, \mathscr{F})$, (2) a probability measure μ on $(\mathcal{X}, \mathscr{F})$, and (3) a map $T : \mathcal{X} \to \mathcal{X}$ that *preserves* μ.
- Definition: If $\varphi : \mathcal{X} \to \mathcal{Y}$ is a measurable map between two measurable spaces, the every (probability) measure μ on $\mathcal{X}$ induces (via φ) a measure $\varphi\mu$ on $\mathcal{Y}$, where $(\varphi\mu)(B) := \mu(\varphi^{-1}B)$ for every measurable $B \subseteq \mathcal{Y}$. Interpretation (when μ is a probability measure): if x is picked at random from $\mathcal{X}$ according to distribution μ, then $\varphi(x)$ is a random element of $\mathcal{Y}$ with distribution $\varphi\mu$.
- Review of some facts from measure theory: semi-algebras and how they make life easier
- Exercise: Every rotation R_α preserves the Lebesgue measure.
- Exercise: The multiplication-by-2 map on $\mathbb{R}/\mathbb{Z}$ preserves the Lebesgue measure.

Wednesday 2022-01-26

- Formulation of Example 3 in terms of a dynamical system (the *Bernoulli shift*).
- In which sense are Example 3 (with $p = 1/2$) and Example 2 equivalent?
- Example 4 (rotation): We have a wheel with an arrow. At every time-step, the wheel rotates by $2\pi\alpha$ radians.

Asymptotically, how often will the arrow point to somewhere inside a given interval I?

- If α is rational, then the asymptotic frequency exists but depends on the initial state.
- Exercise: What are the possible values of the asymptotic frequency when $\alpha = \ell/m$ is rational?
- Kronecker's Theorem: If α is irrational, then the orbit of every point eventually enters every open interval.
- Weyl's Theorem: If α is irrational, then the asymptotic frequency of times at which the arrow points to somewhere inside given interval I is proportional to the length of I, irrespective of the initial state.

Monday 2022-01-24

- Example 2 (Normal numbers): What is the density of 1s in the binary expansion of a "typical" $x \in [0, 1)$?
- Exercise: The binary expansion of a real number $x \in [0, 1)$ is eventually periodic if and only if x is rational.
- Remark: All real numbers in $[0, 1)$ have unique binary expansions except those that are rational with a power of two as the denominator.
- Borel's theorem (without proof): Almost every $x \in [0, 1)$ is normal.
- Example 3 (Bernoulli shift): What is the frequency of 1s in an infinite sequence of independent Bernoulli RVs with parameter p?
- Law of Large Numbers (Weak and Strong versions)

Week 0

Friday 2022-01-21

- Formulation of discrete-time and continuous-time dynamical systems in terms (semigroups of) transformations of the space of all possible states
- Ergodic Theory is (primarily) about the "statistical properties" of the orbits of dynamical systems.
- Example 1 (Billiard): A hard ball moving on a friction-less billiard table
- Sinai's theorem (informal and inaccurate statement): Assuming the table has a a certain property, for almost every starting position and velocity, the fraction of times at which the ball is inside any given region B on the table is proportional to the area of B.