

American University of Beirut
Introduction to Ergodic Theory

MATH 307K – Spring 2022

Assignment 3 (for Friday, March 18)

Problem 1 (Induced isometries). Let μ be a probability measure on a measurable space $\mathcal{X}$ and $T: \mathcal{X} \rightarrow \mathcal{X}$ a measurable map that preserves μ . Verify that for every $1 \leq p \leq \infty$, the map $f \mapsto f \circ T$ is a positive linear isometry of $L^p_\mu(\mathcal{X})$.

Problem 2 (Property of conditional expectation). Let $(\mathcal{X}, \mathcal{F})$ be a measurable space and $T: \mathcal{X} \rightarrow \mathcal{X}$ a measurable map. Prove that

$$\mu(f|_{\mathcal{A}}) \circ T = \mu(f \circ T|_{T^{-1}\mathcal{A}}) \quad \mu\text{-a.e.}$$

for every probability measure μ on $\mathcal{X}$, every function $f \in L^1_\mu(\mathcal{X})$ and every sub- σ -algebra $\mathcal{A} \subseteq \mathcal{F}$.

Problem 3 (L^1 ergodic theorem). Use the L^1 decomposition lemma to give a proof of the L^1 ergodic theorem. *Hint*: Mimic the proof of von Neumann's theorem.

Problem 4 (Characterization of I_1). Let μ be a probability measure on a measurable space $(\mathcal{X}, \mathcal{F})$ and $T: \mathcal{X} \rightarrow \mathcal{X}$ a measurable map that preserves μ . Recall the notation

$$\mathcal{I}_\mu := \{E \in \mathcal{F} : \mu(E \triangle T^{-1}E) = 0\}, \quad I_1 := \{g \in L^1_\mu(\mathcal{X}) : g \circ T = g \text{ } \mu\text{-a.e.}\}.$$

- (a) Verify that $\mathcal{I}_\mu$ is a sub- σ -algebra of $\mathcal{F}$.
- (b) Prove that $f \in I_1$ if and only if $\mu(f|_{\mathcal{I}_\mu}) = f$ μ -a.e.

Problem 5 (Limit in Birkhoff's theorem). Without relying on the decomposition lemma, verify that the function $\bar{f}$ in the original statement of Birkhoff's theorem is $\mu(f|_{\mathcal{I}_\mu})$.

Problem 6 (Weak mixing: irrational rotation). Let $\mathcal{X} := \mathbb{R}/\mathbb{Z}$. Let α be an irrational, and let $R_\alpha := x + \alpha \pmod{1}$ be the *rotation-by- α* map on $\mathcal{X}$. Prove that R_α is not weakly mixing.

Problem 7 (Cartesian product: ergodicity and mixing). Recall that the (Cartesian) product of two measure-preserving dynamical systems $(\mathcal{X}_1, \mathcal{F}_1, \mu_1, T_1)$ and $(\mathcal{X}_2, \mathcal{F}_2, \mu_2, T_2)$ is the system

$$(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \times \mu_2, T_1 \times T_2),$$

where $\mathcal{F}_1 \otimes \mathcal{F}_2$ is the product σ -algebra, $\mu_1 \times \mu_2$ is the product measure, and $(T_1 \times T_2)(x_1, x_2) := (T_1(x_1), T_2(x_2))$. We saw that the product of two weakly mixing systems is again weakly mixing.

- (a) Give an example of two ergodic systems the product of which is not ergodic.
- (b) Is the product of two strongly mixing systems strongly mixing?
- (c) Prove that the product of an ergodic system and a weakly mixing system is ergodic.

Problem 8 (Covergence in density vs. statistical convergence). Let $(\mathcal{X}, \rho)$ be a metric space. Prove that for every $a_0, a_1, a_2, \dots \in \mathcal{X}$ and $a \in \mathcal{X}$, the following conditions are equivalent:

- (i) (statistical convergence) For every $\varepsilon > 0$, we have $\lim_{n \rightarrow \infty} \frac{|\{k < n : \rho(a_k, a) \geq \varepsilon\}|}{n} = 0$.
- (ii) (convergence in density) There exists a set $J \subseteq \mathbb{N}$ of full density such that $\lim_{J \ni n \rightarrow \infty} a_n = a$.

Problem 9 (Counter-example). Find a counter-example to Theorem 1.20 of Walters if we remove the *boundedness* condition. Namely, find an unbounded sequence $a_0, a_1, a_2, \dots$ of non-negative real numbers such that

- $a_n \rightarrow 0$ as $J \ni n \rightarrow \infty$ for some full density set $J \subseteq \mathbb{N}$, but
- $\frac{1}{n} \sum_{k=0}^{n-1} a_k \not\rightarrow 0$ as $n \rightarrow \infty$.

(bonus) **Problem 10** (Weak and strong mixing). Are the following statements true?

- (a) A measure-preserving dynamical system $(\mathcal{X}, \mathcal{F}, \mu, T)$ is strongly mixing if and only if for every $A, B \in \mathcal{F}$ with $\mu(A), \mu(B) > 0$, there exists an $n \in \mathbb{N}$ such that $\mu(A \cap T^{-n}B) > 0$ for all $n \geq n_0$.
- (b) A measure-preserving dynamical system $(\mathcal{X}, \mathcal{F}, \mu, T)$ is weakly mixing if and only if for every $A, B \in \mathcal{F}$ with $\mu(A), \mu(B) > 0$, there exists a set $J \subseteq \mathbb{N}$ of full density such that $\mu(A \cap T^{-n}B) > 0$ for all $n \in J$.