

American University of Beirut
Introduction to Ergodic Theory
 MATH 307K (2026–2027 Fall)

Assignment 1

Problem 1 (Binary expansions). Prove that the binary expansion of a real number $x \in [0, 1)$ is eventually periodic if and only if x is rational.

Problem 2 (Rational rotation). Let $R_\alpha : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ denote the *rotation-by- α* transformation given by $R_\alpha(x) := x + \alpha \pmod{1}$. Suppose that $\alpha = \ell/m$ is a rational ($\ell \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$ are relatively prime). During the lecture, we saw that for every interval $(a, b) \subseteq \mathbb{R}/\mathbb{Z}$ and every point $x \in \mathbb{R}/\mathbb{Z}$, the asymptotic frequency

$$d_{(a,b)}(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{(a,b)}(R_\alpha^k(x))$$

of time steps at which the orbit of x enters (a, b) exists. What are the possible values of $d_{(a,b)}(x)$ when we vary x ?

Problem 3 (Invariant measures: Rotation). Let R_α denote the *rotation-by- α* map as in the previous problem.

- (a) Prove that irrespective of the value of $\alpha \in \mathbb{R}$, the map R_α preserves the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$.
- (b) Can you find another probability measure on $\mathbb{R}/\mathbb{Z}$ that is preserved by R_α ?

Hint: Consider the cases in which α is rational and irrational separately. You may use Weyl's theorem if needed.

Problem 4 (Invariant measures: Multiplication by 2). Let $T : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ denote the *multiplication-by-2* transformation given by $T(x) := 2x \pmod{1}$. Prove that T preserves the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$.

Problem 5 (Bernoulli shift). Let Γ be a finite alphabet, and $\mathcal{X} := \Gamma^{\mathbb{N}}$ the space of all infinite sequences $x = x_0x_1x_2 \cdots$ of symbols from Γ . A *cylinder set* in $\mathcal{X}$ is a set of the form

$$[u] := \{x \in \mathcal{X} : x_0x_1 \cdots x_{\ell-1} = u_0u_1 \cdots u_{\ell-1}\}$$

where $u = u_0u_1 \cdots u_{\ell-1} \in \Gamma^\ell$ is a *word* of length $\ell \in \mathbb{N}$ on Γ . The *product σ -algebra* on $\mathcal{X}$ is the σ -algebra generated by the cylinder sets.

- (a) Verify that, together with the empty set, the cylinder sets form a semi-algebra.

The *product topology* on $\mathcal{X}$ is the topology generated by the cylinder sets.

- (b) Verify that, in the product topology, the cylinder sets are both open and closed.
- (c) Verify that the product topology on $\mathcal{X}$ is compact. *Hint:* Tychonoff's theorem.
- (d) Verify that the product σ -algebra on $\mathcal{X}$ coincides with the Borel σ -algebra corresponding to the product topology.

Let $p : \Gamma \rightarrow [0, 1]$ be a probability distribution on Γ (i.e., a function satisfying $\sum_{a \in \Gamma} p(a) = 1$).

- (e) Show that there exists a unique probability measure π_p on $\mathcal{X}$ such that

$$\pi_p([u]) = \prod_{k=0}^{\ell-1} p(u_k)$$

for every word $u = u_0u_1 \cdots u_{\ell-1} \in \Gamma^\ell$ with $\ell \in \mathbb{N}$. This is called the *Bernoulli measure* (a.k.a. the *product measure*) with marginal p .

Hint: Verify that π_p defined as above is finitely additive on the semi-algebra of cylinder sets and argue that countable additivity follows trivially.

Let $\sigma : \mathcal{X} \rightarrow \mathcal{X}$ be the *shift map* defined by $\sigma(x)_k := x_{k+1}$.

- (f) Verify that the shift map preserves every Bernoulli measure on $\mathcal{X}$.

Problem 6 (Finite dynamical systems). Let $\mathcal{X} := \{0, 1, 2, 3, 4\}$ and consider the map $T : \mathcal{X} \rightarrow \mathcal{X}$, where $T(0) := 1$, $T(1) := 2$, $T(2) := 3$, and $T(3) := 1$, $T(4) := 4$.

- (a) Identify all the probability measures on $\mathcal{X}$ that are preserved by T .
- (b) Prove that for every function $f : \mathcal{X} \rightarrow \mathbb{R}$ and every point $x \in \mathcal{X}$, the limit

$$\bar{f}(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x))$$

exists, and find its value.

Problem 7 (Proof of Poincaré's recurrence theorem). Prove the second claim in Poincaré's theorem. Namely, let μ be a probability measure on a measurable space $\mathcal{X}$ and $T : \mathcal{X} \rightarrow \mathcal{X}$ a measurable map that preserves μ . Let $A \subseteq \mathcal{X}$ be a measurable set with $\mu(A) > 0$. Prove that the orbit of μ -a.e. $x \in A$ returns to A *infinitely many times*.

Hint: Either adapt the proof of the 1st claim, or use the 1st claim as a lemma.