Full Name: ______________________

Student No: ______________________

Grade:

Read before you start:

- Please make sure you write your full name and student number.
- The exam consists of 7 questions, most with multiple parts, and a total score of 145 points.
- All answers require justifications. To get full credit, the justifications must be clearly written, with correct usage of mathematical notations.
- The duration of the exam is 2 hours.

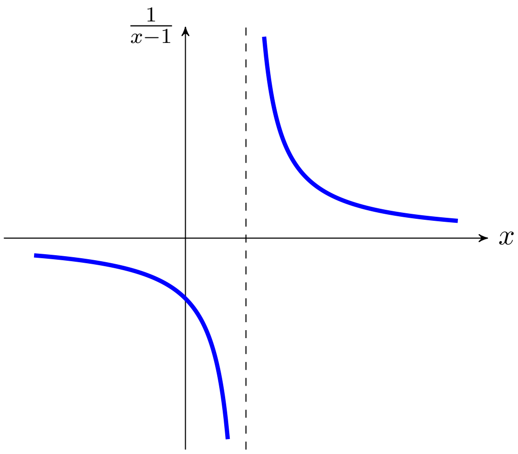
You can use the remainder of this page as scratch paper.

1. (40 points) Determine which of the following statements is True and which is False. In each case, give a short justification.

FALSE The function $f(x) = \dfrac{1}{x-1}$ is decreasing on its domain because $f'(x) = -\dfrac{1}{(x-1)^2}$ is negative.

Solution: The function is clearly not decreasing on its domain. For instance, $0 < 2$ but $f(0) = -1 < 1 = f(2)$.

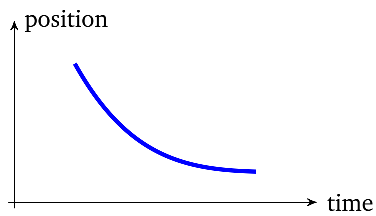
This does not contradict the fact that the derivative of $f(x)$ is negative on its domain because the domain of $f(x)$ is not an interval. Indeed, since $f(x)$ is not defined at $x = 1$, its derivative does not impose any relationship between the values of $f(x)$ for $x < 1$ and the values of $f(x)$ for $x > 1$.



TRUE If a car is going backwards on a road but it is slowing down, then the graph of its position as a function of time is concave up.

Solution: Slowing down while going backwards means that the acceleration of the car (i.e., the second derivative of the position) is positive. Therefore, the position of the car as a function of time is concave up.

This can also be seen visually by imagining how the graph of its position as a function of time must look like.

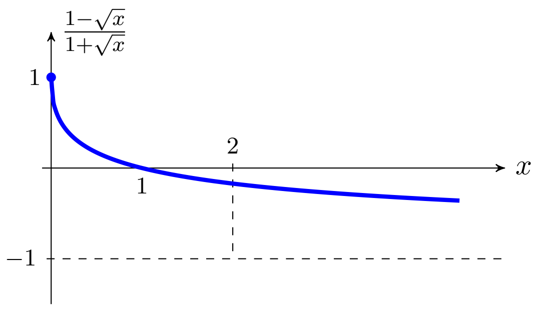


TRUE The function $f(x) = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$ has no horizontal tangent line.

Solution: A function has a horizontal tangent line where its derivative is zero. Let us compute the derivative of $f(x)$:

$$f'(x) = \frac{-\frac{1}{2\sqrt{x}}(1+\sqrt{x}) - (1-\sqrt{x})\frac{1}{2\sqrt{x}}}{(1+\sqrt{x})^2} = \frac{-1}{\sqrt{x}(1+\sqrt{x})^2}\,.$$

Since $f'(x) = 0$ no solution, we conclude that the graph of $f(x)$ has no horizontal tangent line.

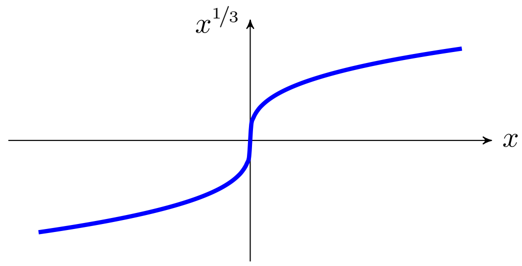


TRUE Since the function $f(x) = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$ is continuous on the interval $[0, 2]$, it necessarily achieves a maximum on this interval.

Solution: This is true by the Extreme Value Theorem. Here, the maximum is achieved at the boundary point $x = 0$.

FALSE Since the function $f(x) = x^{1/3}$ is continuous everywhere, it is also differentiable everywhere.

Solution: Continuity does not imply differentiability, and the function $f(x) = x^{1/3}$ is a witness to this: while $f(x)$ is continuous everywhere, it is not differentiable at $x = 0$.



FALSE $\dfrac{\mathrm{d}}{\mathrm{d}x}\pi^{13} = 13\pi^{12}$.

Solution: π^{13} is a constant; its derivative is 0.

TRUE $\lim_{h\to 0} \dfrac{\sin(2x+h) - \sin(2x)}{h} = \cos(2x)$.

Solution: The left-hand side is, by definition, the derivative of $f(y) = \sin(y)$ at point $y = 2x$. We know that $f'(y) = \cos(y)$, hence $f'(2x) = \cos(2x)$.

FALSE Since $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$, it follows that $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin(2x) = \cos(2x)$.

Solution: By the chain rule, $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin(2x) = 2\cos(2x)$.

2. (10 points) Find the derivative of the following functions:

(a) $f(x) = \dfrac{x}{\sqrt{2-x}}$

Solution: We have

$$\begin{aligned} f'(x) &= \frac{\sqrt{2-x} - x\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{2-x}}{(\sqrt{2-x})^2} && \text{(by the quotient rule)} \\ &= \frac{\sqrt{2-x} - x\frac{-1}{2\sqrt{2-x}}}{(\sqrt{2-x})^2} && \text{(by the chain rule)} \\ &= \boxed{\frac{4-x}{2(2-x)\sqrt{2-x}}} && \text{(simplification)} \end{aligned}$$

(b) $g(x) = \sin(3x)\cos(3x)$

Solution: We have

$$\begin{aligned} g'(x) &= 3\cos(3x)\cos(3x) + \sin(3x)(-3\sin(3x)) && \text{(by the product rule)} \\ &= \boxed{3\cos(3x)^2 - 3\sin(3x)^2} \end{aligned}$$

Alternatively, using the trigonometric identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, we note that $g(x) = \frac{1}{2}\sin(6x)$. Therefore, $g'(x) = \boxed{3\cos(6x)}$.
The trigonometric identity $\cos(2\theta) = \cos(\theta)^2 - \sin(\theta)^2$ ensures that the two answers are the same.

3. (5 points) Use the definition to calculate the derivative of $f(x) = \sqrt{x+1}$ at $x = 3$.
[*Note*: You will receive no points for using derivative rules, but you may use them to check your answer.]

Solution: We have

$$\begin{aligned} f'(3) &= \lim_{x\to 3} \frac{\sqrt{x+1} - \sqrt{3+1}}{x-3} && \text{(by definition)} \\ &= \lim_{x\to 3} \frac{\sqrt{x+1} - 2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} && \text{(multiplying and dividing by a non-zero value)} \\ &= \lim_{x\to 3} \frac{x+1-4}{(x-3)\sqrt{x+1}+2} && \text{(expansion)} \\ &= \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2} && \text{(cancellation of non-zero common factors)} \\ &= \frac{1}{\lim_{x\to 3}(\sqrt{x+1}+2)} && \text{(arithmetic rule of limits)} \\ &= \boxed{1/4} && \text{(arithmetic rule of limits)} \end{aligned}$$

To double-check our answer, we note that, using differentiation rules, we have $f'(x) = \frac{1}{2\sqrt{x+1}}$, which evaluates to $1/4$ at $x = 3$.

4. (10 points) Find all the points on the curve described by the equation $x^2 + y^2 + xy = 3$ at which the tangent line is horizontal.

Solution: Implicit differentiation with respect to x gives

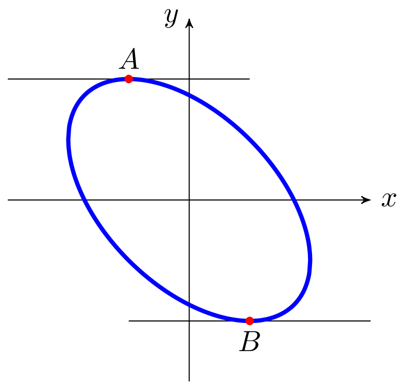
$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} + y + x\frac{\mathrm{d}y}{\mathrm{d}x} = 0\ . \tag{☼}$$

Points with horizontal tangent lines are precisely those points on the curve at which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Plugging this in (☼), we obtain that such points satisfy $2x + y = 0$. Of course, the points with horizontal tangent lines also satisfy the equation of the curve. Therefore, such points are the solutions of the following system of two equations:

$$\begin{cases} x^2 + y^2 + xy = 3\ , \\ 2x + y = 0 \end{cases}$$

Solving this system, we obtain two solutions:

$$A : \boxed{(x, y) = (-1, 2)} \qquad \text{and} \qquad B : \boxed{(x, y) = (1, -2)}$$



5. (10 points) Assuming $f(2) = 5$ and $f'(2) = 11$, compute the following:

(a) $\left.\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{f(x)}{x^2}\right)\right|_{x=2}$

Solution: Using the quotient rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{x^2}\right) = \frac{f'(x)\cdot x^2 - f(x)\cdot(2x)}{(x^2)^2} = \frac{xf'(x) - 2f(x)}{x^3} .$$

Substituting $x = 2$, we get

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{x^2}\right)\right|_{x=2} = \frac{2\times 11 - 2\times 5}{2^3} = \boxed{3/2} .$$

(b) $\left.\dfrac{\mathrm{d}}{\mathrm{d}x}f(\sqrt{x})\right|_{x=4}$

Solution: Using the chain rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}f(\sqrt{x}) = \frac{1}{2\sqrt{x}}f'(\sqrt{x}) .$$

Substituting $x = 4$, we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}f(\sqrt{x})\right|_{x=4} = \frac{1}{2\sqrt{4}}f'(\sqrt{4}) = \frac{1}{4}\times 11 = \boxed{11/4} .$$

6. (20 points) Consider the function $f(x) = x^5 + 4x + cos(\pi x)$.

(a) Show that $f(x)$ is an increasing function. [*Hint*: $\sin(\pi x) \geq -1$.]

Solution: The function $f(x)$ is defined everywhere, and is differentiable. Its derivative is

$$f'(x) = 5x^4 + 4 - \pi \sin(\pi x) .$$

The term $5x^4$ is always non-negative. Since $\sin(\pi x) \leq 1$, we have $\pi \sin(\pi x) \leq \pi < 4$, hence $4 - \pi \sin(\pi x) > 0$ for every x. Therefore, $f'(x) > 0$ for every x. We conclude that $f(x)$ is increasing.

(b) Argue that $f(x)$ has an inverse. [*Hint*: Use the MVT to argue that $y = f(x)$ uniquely determines x.]

Solution: To show that $f(x)$ has an inverse, we need to show that $y = f(x)$ uniquely determines x.
Suppose this is not the case. This means there are two distinct points x_1 and x_2 with $f(x_1) = f(x_2)$.
Note that $f(x)$ is continuous and differentiable everywhere. Therefore, by the Mean Value Theorem (or Rolle's theorem), there exists a point c strictly between x_1 and x_2 such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 .$$

But this contradicts what we have found in part (a), namely that $f'(x)$ is always strictly positive. We conclude that no such values x_1, x_2 exist, which means $f(x)$ has an inverse.

Let $g(y)$ denote the inverse of $f(x)$.

(c) Show that $g(4) = 1$.

Solution: Since $f(1) = 1^5 + 4 \times 1 + \cos(\pi) = 1 + 4 - 1 = 4$, it immediately follows that $g(4) = 1$.

(d) Find $g'(4)$. [*Hint*: Set $y = f(x)$, and use implicit differentiation with respect to y to find $\left.\frac{\mathrm{d}x}{\mathrm{d}y}\right|_{\substack{x=1\\y=4}}$.]

Solution: Following the hint, let $y = f(x)$, or equivalently $x = g(y)$. We are looking for $g'(y) = {}^{\mathrm{d}x}/_{\mathrm{d}y}$ evaluated at $y = 4$, or equivalently, at $x = 1$.
Implicit differentiation of the equation

$$y = x^5 + 4x + cos(\pi x) .$$

with respect to y gives

$$1 = 5x^4 \frac{\mathrm{d}x}{\mathrm{d}y} + 4\frac{\mathrm{d}x}{\mathrm{d}y} - \pi \sin(\pi x)\frac{\mathrm{d}x}{\mathrm{d}y} .$$

Solving for ${}^{\mathrm{d}x}/_{\mathrm{d}y}$, we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{5x^4 + 4 - \pi \sin(\pi x)} .$$

Substituting $y = 4$, or equivalently, $x = 1$, we obtain

$$g'(4) = \left.\frac{\mathrm{d}x}{\mathrm{d}y}\right|_{x=1} = \frac{1}{5 + 4 - \pi \sin(\pi)} = \boxed{1/9} .$$

7. (50 points) Let $f(x) = (x+4)(x-1)(x-3) = x^3 - 13x + 12$.

(a) Identify the domain of $f(x)$.

> **Solution:** The function $f(x)$ has a well-defined value for every choice of x, hence the domain of $f(x)$ is $(-\infty, \infty)$.

(b) Identify the x-intercepts and the y-intercepts of the graph of the function.

> **Solution:**
> x-intercepts: Setting $f(x) = 0$, we find that the graph hits the x-axis at three points
>
> $$\boxed{x=-4} \qquad \boxed{x=1} \qquad \boxed{x=3}$$
>
> y-intercept: Setting $x = 0$, we find that the graph hits the y-axis at $\boxed{y = 12}$.

(c) Identify the intervals over which $f(x)$ is positive and the intervals over which it is negative.

> **Solution:** We can analyse the sign of $f(x)$ by analysing the signs of its three factors $(x+4)$, $(x-1)$, and $(x-3)$:

	$x < -4$	$x = -4$	$-4 < x < 1$	$x = 1$	$1 < x < 3$	$x = 3$	$3 < x$
$x-3$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x+4$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

> In summary,

$f(x)$:	$-$	0	$+$	0	$-$	0	$+$	
		-4		1		3		x

(d) Does the graph of $f(x)$ have any symmetries?

> **Solution:** There are no "standard" symmetries: $f(x)$ is not odd, even, or periodic.
> However, we may notice that the graph of $f(x)$ is the same as the graph of $g(x) = x^3 - 13x$ shifted upwards by 12 units. Since $g(x)$ is odd, its graph is symmetric about the origin. It follows that the graph of $f(x)$ is symmetric about the point $(0, 12)$.

(e) Does the graph of $f(x)$ have any vertical or horizontal asymptotes? If so, identify them.

> **Solution:**
> Vertical asymptotes: Since $f(x)$ is continuous everywhere, it has a (finite) limit at every point. This means the graph of $f(x)$ has no vertical asymptotes.
> Horizontal asymptotes: Since
>
> $$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^3\left(1 - \frac{13}{x^2} + \frac{12}{x^3}\right) = +\infty\ ,$$
> $$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3\left(1 - \frac{13}{x^2} + \frac{12}{x^3}\right) = -\infty\ ,$$
>
> the graph of $f(x)$ has no horizontal asymptotes.

(f) Find the critical points of $f(x)$.

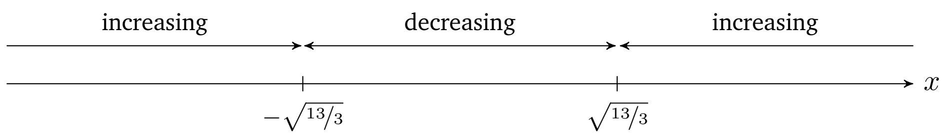
Solution: The *critical points* of $f(x)$ are the points at which $f'(x)$ is either 0 or undefined. The derivative of $f(x)$ is $f'(x) = 3x^2 - 13$, which is clearly defined everywhere. Setting $f'(x) = 0$, we find two critical points at $\boxed{x = \pm\sqrt{13/3}} \approx 2.08$.

(g) Identify the intervals over which $f(x)$ is increasing and the intervals over which it $f(x)$ is decreasing.

Solution: Since $f(x)$ is defined and differentiable everywhere, its monotonicity on different intervals is determined by the sign of its derivative.
Note that $f'(x) = \left(x + \sqrt{13/3}\right)\left(x - \sqrt{13/3}\right)$. We can analyse the sign of $f'(x)$ by analysing the signs of its factors $\left(x + \sqrt{13/3}\right)$ and $\left(x - \sqrt{13/3}\right)$:

	$x < -\sqrt{13/3}$	$x = -\sqrt{13/3}$	$-\sqrt{13/3} < x < \sqrt{13/3}$	$x = \sqrt{13/3}$	$\sqrt{13/3} < x$
$x - \sqrt{13/3}$	$-$	$-$	$-$	0	$+$
$x + \sqrt{13/3}$	$-$	0	$+$	$+$	$+$
$f(x)$	$+$	0	$-$	0	$+$

In summary,



(h) Identify the intervals over which $f(x)$ is concave up and the intervals over which it is concave down.

Solution: Since $f(x)$ is twice-differentiable everywhere, its concavity on different intervals is determined by the sign of its second derivative.
The second derivative of $f(x)$ is $f''(x) = 6x$, which is positive for $x > 0$ and negative for $x < 0$. Therefore, $f(x)$ is concave up on $(0, +\infty)$ and concave down on $(-\infty, 0)$.

(i) Does the graph of $f(x)$ have any inflection point? If so, identify them.

Solution: An *inflection point* is a point where the graph of $f(x)$ has a tangent line and the concavity of $f(x)$ changes.
The concavity of $f(x)$ changes at $x = 0$. Since $f(x)$ is differentiable at $x = 0$, it has a tangent line at that point. We conclude that $x = 0$ is an inflection point.

(j) Sketch the graph of $f(x)$, indicating the information gathered above.

Solution: