Full Name: ______________________

Student No: ______________________

Grade:

Read before you start:

- Please make sure you write your full name and student number.
- The exam consists of 10 questions, each with multiple parts, and a total score of 140 points.
- All answers require justifications. To get full credit, the justifications must be clearly written, with correct usage of mathematical notations.
- The duration of the exam is 2 hours.

You can use the remainder of this page as scratch paper.

1. (35 points) Determine which of the following statements is True and which is False. In each case, give a short justification.

TRUE The two functions $f(x) = x^2$ and $g(u) = u^2$ are one and the same.

Solution: What makes a function is how it transforms each input into a corresponding output. What name we give to the input is irrelevant.

FALSE The two functions $f(x) = \dfrac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$ are one and the same.

Solution: The two functions have different domains. While $g(x)$ is defined for every real number x, the function $f(x)$ is not defined when $x = -1$.

TRUE Whether or not $\lim_{x\to 1} f(x)$ exists does not depend on how $f(1)$ is defined.

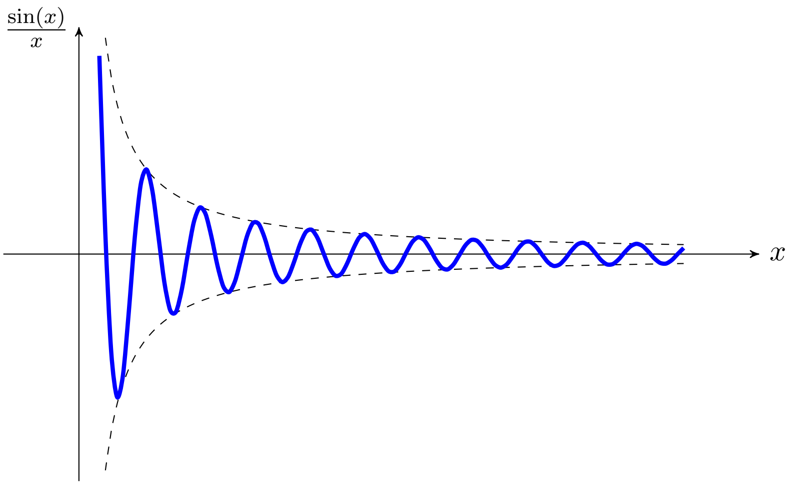
Solution: Whether or not $\lim_{x\to 1} f(x)$ exists depends only on $f(x)$ for values of x that are close to 1 but different from it.

FALSE If $\lim_{x\to a} f(x) = 0$ and g is defined near a, then $\lim_{x\to a} f(x)g(x) = 0$.

Solution: As a counter-example, suppose $f(x) = x$ and $g(x) = 1/x$.

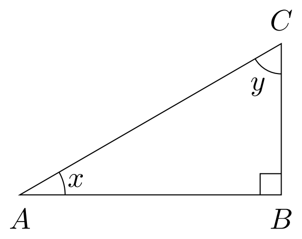
TRUE A function can cross its horizontal asymptote.

Solution: As an example, the function $f(x) = \dfrac{\sin(x)}{x}$ crosses its horizontal asymptote $y = 0$ infinitely many times.



TRUE $\cos(\pi/2 - x) = \sin x$ for every x.

Solution: This can be verified in at least two different ways.
Method 1 (the special case in which $0 < x < \pi/2$):



Consider a triangle $\triangle ABC$ in which $\angle ABC$ is a right angle and $\angle CAB = x$. Let $y = \angle BCA$. Then, clearly $y = \pi/2 - x$, hence

$$\cos(\pi/2 - x) = \sin(y) = \frac{AB}{AC} = \cos(x) \ .$$

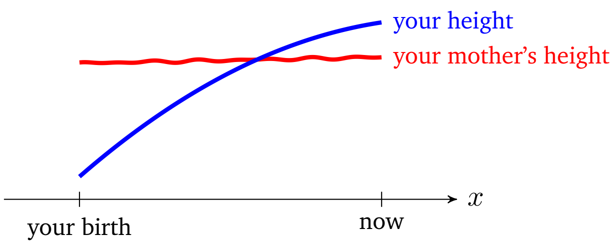
Method 2 (using the identity $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$):
Choosing $\alpha = \pi/2$ and $\beta = -x$, we get

$$\begin{aligned}\cos(\pi/2 - x) &= \cos(\pi/2)\cos(-x) - \sin(\pi/2)\sin(-x) \\ &= 0 \times \cos(-x) - 1 \times \sin(-x) = -\sin(-x) = \sin(x) \ .\end{aligned}$$

This works for every x.

TRUE If you are currently taller than your mother, then at some time since your birth, you and your mother must have had exactly the same height.

Solution:



The validity of this statement is intuitively clear, but can also be argued using the *intermediate value theorem*. Indeed, let $h_{\text{you}}(t)$ denote your height and $h_{\text{mom}}(t)$ denote your mother's height as functions of time. Consider the difference

$$d(t) = h_{\text{you}}(t) - h_{\text{mom}}(t) \ .$$

Clearly, $d(t)$ is a continuous function because $h_{\text{you}}(t)$ and $h_{\text{mom}}(t)$ are. Furthermore,

- $d(\text{your birth}) < 0$ (at the time of your birth, your mother was taller than you),
- $d(\text{now}) > 0$ (you are currently taller than your mother).

Therefore, by the intermediate value theorem, there must have been a time t^* between your birth and now such that $d(t^*) = 0$ (i.e., you and your mother had the same height).

2. (10 points)

(a) Can an even function have an inverse?
[*Recall*: The *inverse* of a function f is a function g such that $y = f(x)$ if and only if $x = g(y)$]

Solution: "No" (but see the disclaimer below).
Let us start with an example of an even function $f(x) = x^2$. In order for f to be invertible, we must be able to recover x if we know $y = f(x)$. However, if $f(1) = f(-1) = 1$, so if $y = 1$, we cannot tell whether $x = 1$ or $x = -1$.
In general, saying that f is even means that $f(-x) = f(x)$ for every x in the domain of f. If $x \neq 0$ and we observe that the output of the function is $y = f(x) = f(-x)$, then we cannot tell whether the input has been x or $-x$. This means that f cannot be invertible.

Disclaimer: Consider a (useless) function $f(x)$ which is defined only at $x = 0$ (i.e., the domain of f is $\{0\}$). Note that f is *technically* an even function and f is *technically* invertible. Thus, if we are meticulous about mathematical definitions, then must admit that an even function can have an inverse.

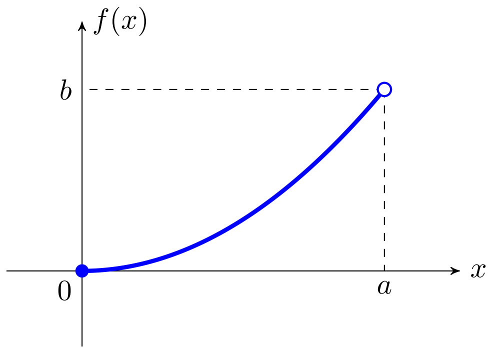
(b) Suppose f is an even function and g is an odd function. Is the function $g \circ f$ even, odd, or neither?

Solution: Since f is even, $f(-x) = f(x)$ for every x in the domain of f. Since g is odd, $g(-y) = -g(y)$ for every x in the domain of g. Hence,

$$(g \circ f)(-x) = g(f(-x)) = g(f(x)) = (g \circ f)(x)$$

in the domain of $g \circ f$, which means $g \circ f$ is even.

3. (10 points) The following is the graph of a function $f(x)$.



(a) Identify the domain and the range of f.

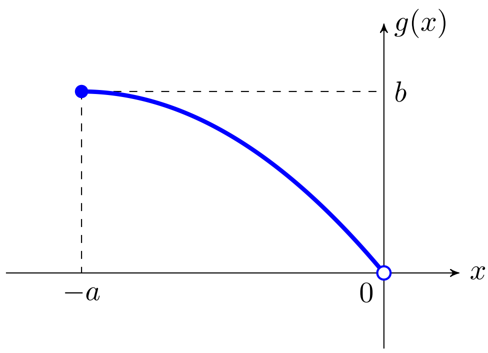
Solution:

domain $= [0, a)$ (all numbers between 0 and a, including 0 and excluding a)

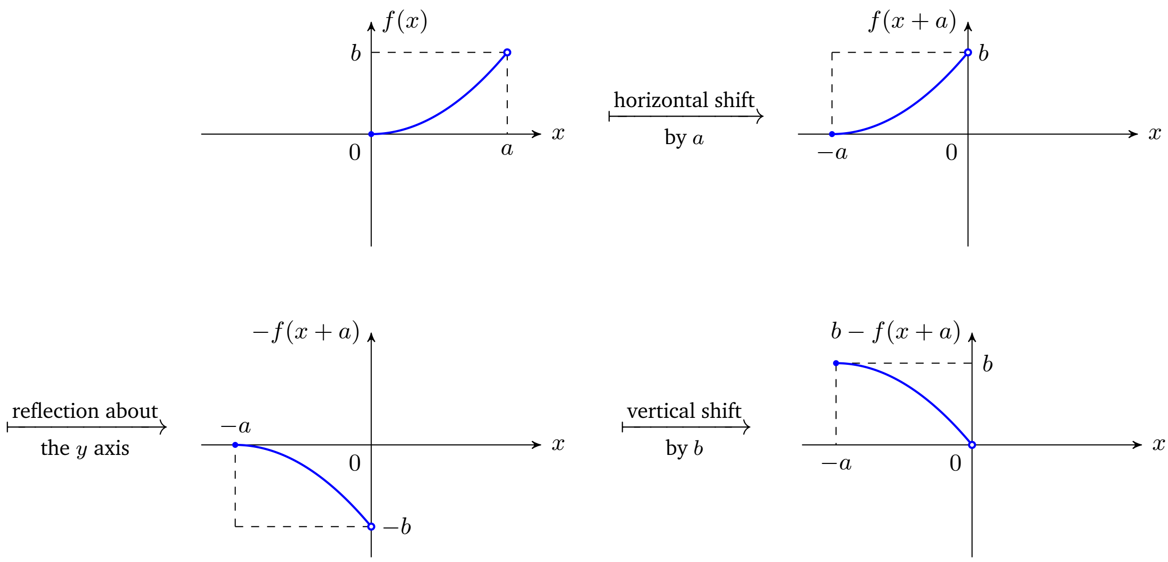
range $= [0, b)$ (all numbers between 0 and b, including 0 and excluding b)

(b) Sketch the graph of the function $g(x) = b - f(x + a)$.

Solution:



Namely,



4. (20 points) Evaluate each of the following limits or determine it does not exist.

(a) $\lim_{x \to -2} \sqrt{2x^3 + 20}$

Solution: Using the arithmetic laws of limits,

$$\lim_{x \to -2} \sqrt{2x^3 + 20} = \sqrt{\lim_{x \to -2}(2x^3 + 20)} = \sqrt{2\Big(\lim_{x \to -2} x\Big)^3 + 20} = \sqrt{2 \times (-2)^3 + 20} = \sqrt{4} = \boxed{2}\,.$$

(b) $\lim_{x \to -5} \dfrac{x^3 - 25x}{x^2 + 5x}$

Solution: Since the limit of the denominator is 0, the arithmetic laws do not directly apply. Instead, we can start by noting that

$$\frac{x^3 - 25x}{x^2 + 5x} = \frac{x(x-5)(x+5)}{x(x+5)} = x - 5 \qquad \text{as long as } x \neq 0 \text{ and } x \neq -5.$$

When taking the limit as $x \to -5$, we are concerned with the values of x that are close to -5 but not equal to -5, so the cancellation is valid. Therefore,

$$\lim_{x \to -5} \frac{x^3 - 25x}{x^2 + 5x} = \lim_{x \to -5}(x - 5) = \boxed{-10}\,.$$

(c) $\lim_{x \to 1} \dfrac{x^2 - x}{|x - 1|}$

Solution: To handle the absolute value in the denominator, we can start by computing the limits from the right and the limit from the left.
When x approaches 1 from the right, we have $|x - 1| = x - 1$, hence

$$\lim_{x \to 1^+} \frac{x^2 - x}{|x - 1|} = \lim_{x \to 1^+} \frac{x(x-1)}{x - 1} = \lim_{x \to 1^+} x = 1\,.$$

When x approaches 1 from the left, we have $|x - 1| = -(x - 1)$, hence

$$\lim_{x \to 1^-} \frac{x^2 - x}{|x - 1|} = \lim_{x \to 1^-} \frac{x(x-1)}{-(x - 1)} = \lim_{x \to 1^-} -x = -1\,.$$

Since the left-limit and the right-limit do not coincide, we conclude that $\lim_{x \to 1} \dfrac{x^2 - x}{|x - 1|}$ does not exist.

(d) $\lim_{x \to 0} \dfrac{3x^2}{(\sin 2x)^2}$

Solution: Using the arithmetic laws of limits, we have

$$\clubsuit = \lim_{x \to 0} \frac{3x^2}{(\sin 2x)^2} = \frac{3}{4} \lim_{x \to 0} \frac{4x^2}{(\sin 2x)^2} = \frac{3}{4} \lim_{x \to 0} \left(\frac{2x}{\sin(2x)} \right)^2 = \frac{3}{4} \left(\lim_{x \to 0} \frac{2x}{\sin(2x)} \right)^2$$

Using the change of variable $y = 2x$ and the fact that $y \to 0$ as $x \to 0$, we have

$$\lim_{x \to 0} \frac{2x}{\sin(2x)} = \lim_{y \to 0} \frac{y}{\sin y} = \frac{1}{\lim_{y \to 0} \frac{\sin y}{y}} = 1 \; .$$

Therefore,

$$\clubsuit = \frac{3}{4} \times 1^2 = \boxed{3/4} \; .$$

5. (10 points) Consider the function

$$f(x) = \begin{cases} \dfrac{\sqrt{x}-1}{x-1} & \text{if } x > 1, \\ 5 & \text{if } x = 1, \\ \dfrac{2x-2}{x^2+2x-3} & \text{if } x < 1. \end{cases}$$

Does $\lim_{x\to 1} f(x)$ exist? If yes, identify its value. If not, explain why.

Solution: Let us start with finding the limit of $f(x)$ as x approaches 1 from the right and from the left. When x approaches 1 from the right, $f(x)$ is defined using the first expression. Therefore,

$$\begin{aligned}
\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} \frac{\sqrt{x}-1}{x-1} \\
&= \lim_{x\to 1^+} \frac{\sqrt{x}-1}{x-1} \times \frac{\sqrt{x}+1}{\sqrt{x}+1} && (\sqrt{x}+1 \neq 0 \text{ when } x \text{ is close to } 1) \\
&= \lim_{x\to 1^+} \frac{x-1}{(x-1)(\sqrt{x}+1)} && (\text{algebraic identity } (a-b)(a+b) = a^2-b^2) \\
&= \lim_{x\to 1^+} \frac{1}{\sqrt{x}+1} && (\text{cancellation is justified because } x \neq 1) \\
&= \frac{1}{2} && (\text{arithmetic laws of limits})
\end{aligned}$$

When x approaches 1 from the left, $f(x)$ is defined using the third expression. Hence,

$$\begin{aligned}
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \frac{2x-2}{x^2+2x-3} \\
&= \lim_{x\to 1^-} \frac{2(x-1)}{(x-1)(x+3)} \\
&= \lim_{x\to 1^-} \frac{2}{x+3} && (\text{cancellation is justified because } x \neq 1) \\
&= \frac{1}{2} && (\text{arithmetic laws of limits})
\end{aligned}$$

Since the left-limit and the right-limit coincide, <u>the limit exists</u> and we have

$$\lim_{x\to 1} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = \boxed{1/2}.$$

6. (10 points) Evaluate each of the following limits or determine it does not exist.

(a) $\displaystyle\lim_{x\to+\infty} \frac{5x^2-2x+1}{5x^2+2x+1}$

Solution: Since neither the numerator nor the denominator has a limit as $x \to +\infty$, the arithmetic laws of limits at infinity do not directly apply. We can write

$$\begin{aligned}
\lim_{x\to+\infty} \frac{5x^2-2x+1}{5x^2+2x+1} &= \lim_{x\to+\infty} \frac{x^2\left(5-\frac{2}{x}+\frac{1}{x^2}\right)}{x^2\left(5+\frac{2}{x}+\frac{1}{x^2}\right)} && \text{(factoring the dominant terms)}\\
&= \lim_{x\to+\infty} \frac{5-\frac{2}{x}+\frac{1}{x^2}}{5+\frac{2}{x}+\frac{1}{x^2}} && \text{(cancellation justified since } x \neq 0\text{)}\\
&= \frac{5-0+0}{5+0+0} && \text{(arithmetic laws of limits at infinity)}\\
&= \boxed{1}\,.
\end{aligned}$$

(b) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^2-(\sin x)^2}}{x+\sin x}$

Solution: Since neither the numerator nor the denominator has a limit as $x \to -\infty$, the arithmetic laws of limits at infinity do not directly apply. We can write

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{x^2-(\sin x)^2}}{x+\sin x} &= \lim_{x\to-\infty} \frac{\sqrt{x^2\left(1-\frac{\sin(x)^2}{x^2}\right)}}{x\left(1+\frac{\sin x}{x}\right)} && \text{(factoring the dominant terms)}\\
&= \lim_{x\to-\infty} \frac{-x\sqrt{1-\frac{\sin(x)^2}{x^2}}}{x\left(1+\frac{\sin x}{x}\right)} && (\sqrt{x^2} = -x \text{ when } x \text{ is negative})\\
&= \lim_{x\to-\infty} \frac{-\sqrt{1-\frac{\sin(x)^2}{x^2}}}{1+\frac{\sin x}{x}} && \text{(cancellation justified since } x \neq 0\text{)}\\
&= \frac{-\sqrt{1-0}}{1+0} && \text{(arithmetic laws and sandwiching)}\\
&= \boxed{-1}\,.
\end{aligned}$$

For the fourth equality, we have used the fact that, when $x < 0$,

$$\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{-1}{x}$$

and $\lim_{x\to-\infty} \dfrac{1}{x} = \lim_{x\to-\infty} \dfrac{-1}{x} = 0$, hence by the *sandwich theorem*, $\displaystyle\lim_{x\to-\infty} \frac{\sin x}{x} = 0$.

7. (15 points) Consider the function $f(x) = \dfrac{x^2 + \sqrt{x+1}}{(x+1)(x+2)}$.

(a) Identify the domain of f.

Solution: In order for the formula for $f(x)$ to make sense, we must have

- $x \geq -1$ (so that $\sqrt{x+1}$ is meaningful),
- $x \neq -1$ and $x \neq -2$ (so that the denominator is non-zero).

Therefore, the domain of $f(x)$ is the set of all values x such that $x > -1$.

(b) Identify the vertical and horizontal asymptotes of f.

Solution: The vertical asymptotes correspond to infinite limits. Note that $x = -2$ is not in the domain of f, hence the only place at which the limit $f(x)$ can *potentially* be infinite is at $x = -1$. The values $x \leq -1$ are not in the domain of f, so let us examine the right-limit of $f(x)$ as $x \to -1^+$, that is,

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{x^2 + \sqrt{x+1}}{(x+1)(x+2)}$$

When $x \to -1^+$, the numerator approaches 1. In the denominator, the factor $x + 2$ approaches 1 but the factor $x + 1$ approaches 0 from the right. Therefore, the limit does not exist and we have

$$\lim_{x \to -1^+} f(x) = +\infty \ .$$

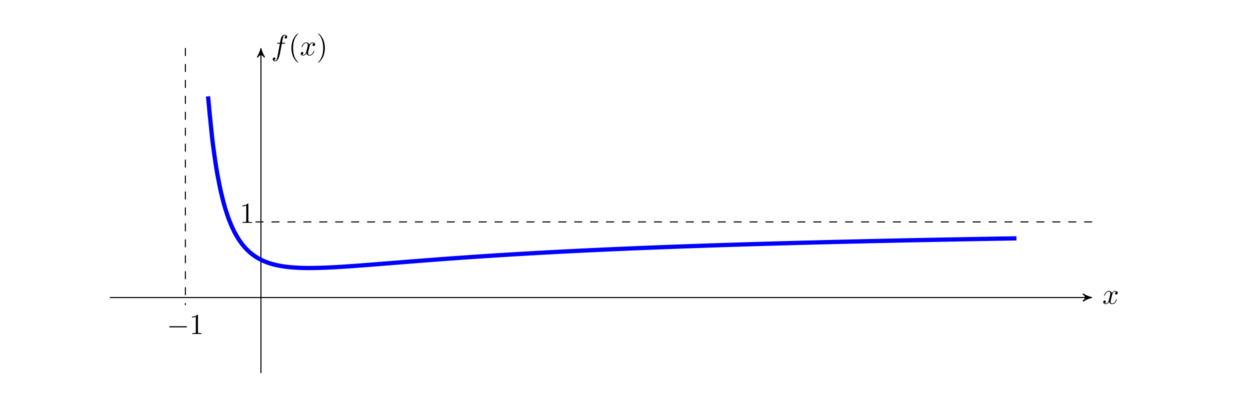
This, in particular, means that <u>the line $x = -1$ is a vertical asymptote of f</u>.

The horizontal asymptotes corresponds to the limits at $+\infty$ and $-\infty$. Since the domain contains only values $x > -1$, we only need to examine the limit as $x \to +\infty$. We have

$$\begin{aligned}
\lim_{x \to +\infty} f(x) &= \lim_{x \to +\infty} \frac{x^2 + \sqrt{x+1}}{(x+1)(x+2)} \\
&= \lim_{x \to +\infty} \frac{x^2 \left(1 + \sqrt{\dfrac{1}{x^3} + \dfrac{1}{x^4}}\right)}{x^2 \left(1 + \dfrac{3}{x} + \dfrac{2}{x^2}\right)} && \text{(factoring the dominant terms)} \\
&= \lim_{x \to +\infty} \frac{1 + \sqrt{\dfrac{1}{x^3} + \dfrac{1}{x^4}}}{1 + \dfrac{3}{x} + \dfrac{2}{x^2}} && \text{(cancellation justified since } x \neq 0\text{)} \\
&= 1 && \text{(arithmetic laws of limits at infinity)}
\end{aligned}$$

This, in particular, means that <u>the line $y = 1$ is a horizontal asymptote of f</u>.

The graph of f is sketched below.



8. (10 points) Consider the function $f(x) = \dfrac{x + \cos x}{2 + \sin x}$.

(a) Argue that f has a zero between $-\pi$ and π.
[*Recall*: A *zero* of a function f is a value a such that $f(a) = 0$.]

Solution: First, note that f is continuous everywhere because it is the quotient of two continuous functions where the denominator is never zero.
Now, observe that

$$f(-\pi) = \frac{-\pi + \cos(-\pi)}{2 + \sin(-\pi)} = \frac{-\pi - 1}{2} < 0 \,, \qquad f(\pi) = \frac{\pi + \cos(\pi)}{2 + \sin(\pi)} = \frac{\pi + 1}{2} > 0 \,.$$

Since f is continuous over $[-\pi, \pi]$, it follows from the *intermediate value theorem* that $f(a) = 0$ for some a satisfying $-\pi \leq a \leq \pi$.

(b) Argue that f has a zero between -1 and 0.

Solution: Note that

$$f(-1) = \frac{-1 + \cos(-1)}{2 + \sin(-1)} < \frac{-1 + 1}{2 - 1} = 0 \,.$$

because $\cos(-1) < 1$ and $\sin(-1) > -1$. Furthermore,

$$f(0) = \frac{0 + \cos 0}{2 + \sin 0} = \frac{1}{2} > 0 \,.$$

Since f is continuous over $[-1, 0]$ and $f(-1) < 0$ and $f(0) > 0$, it follows from the *intermediate value theorem* that $f(b) = 0$ for some b satisfying $-1 \leq b \leq 0$.

9. (10 points) Consider the function $f(x) = x\lfloor x \rfloor$.
[*Recall*: $\lfloor x \rfloor$ stands for the *floor* of x, that is, what we get if we round x down.]

(a) Argue that f is continuous at $x = 0$.
[*Hint*: Use the Sandwich Theorem.]

Solution: Note that $x - 1 < \lfloor x \rfloor \leq x$ for every x. It follows that

$$x(x-1) \leq x\lfloor x \rfloor \leq x^2 \qquad \text{for } x > 0,$$
$$x(x-1) \geq x\lfloor x \rfloor \geq x^2 \qquad \text{for } x < 0.$$

Since $\lim_{x\to 0} x(x-1) = \lim_{x\to 0} x^2 = 0$, it follows from the *sandwich theorem* that $\lim_{x\to 0} f(x) = 0$. Furthermore, $f(0) = 0\lfloor 0 \rfloor = 0$. Since the limit as $x \to 0$ exists and coincides with the value of the function at $x = 0$, we conclude that f is continuous at $x = 0$.

(b) Identify all the points at which $f(x)$ is continuous.

Solution: If n is any integer other than 0, we have

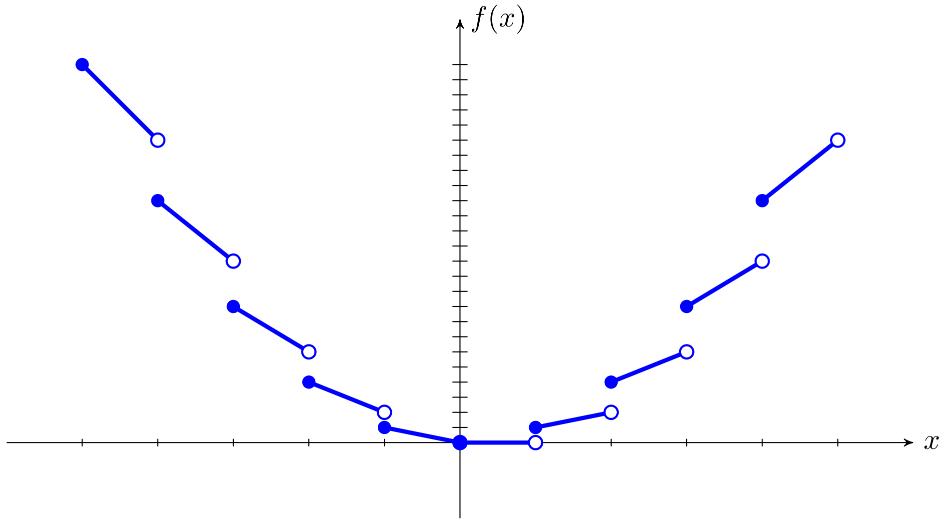
$$\lim_{x\to n^-} f(x) = \lim_{x\to n^-} x\lfloor x \rfloor = n(n-1) \ ,$$
$$\lim_{x\to n^+} f(x) = \lim_{x\to n^+} x\lfloor x \rfloor = n^2$$

Since $n(n-1) \neq n^2$ when $n \neq 0$, we find that $f(x)$ is not continuous at $x = n$.
On the other hand, if a is any real number that is not an integer, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} x\lfloor x \rfloor = \lim_{x\to a} x\lfloor a \rfloor = a\lfloor a \rfloor \ ,$$

which coincides with $f(a)$. Hence, $f(x)$ is continuous at $x = a$.
In summary, the points at which $f(x)$ is continuous are precisely the non-integers and 0.

The graph of $f(x)$ is sketched below.



10. (10 points) Consider the function $f(x) = \dfrac{2-x}{x-1}$. We know that $\lim_{x\to\infty} f(x) = -1$.

(a) How large must x be in order to guarantee that $f(x) = -1 \pm 0.01$?

Solution: We would like to have $\left|f(x) - (-1)\right| < 0.01$. Observe that

$$\left|f(x) - (-1)\right| = \left|\frac{2-x}{x-1} - (-1)\right| = \left|\frac{2-x+x-1}{x-1}\right| = \left|\frac{1}{x-1}\right|.$$

The latter is less than 0.01 if $|x-1| > \dfrac{1}{0.01} = 100$, in particular, if $x > 101$. Therefore, $f(x)$ is guaranteed to be -1 ± 0.01 as long as $x > 101$.

(b) Given a generic $\varepsilon > 0$, find $M > 0$ such that $|f(x) - (-1)| < \varepsilon$ whenever $x > M$.

Solution: As observed in the previous part,

$$\left|f(x) - (-1)\right| = \left|\frac{1}{x-1}\right|.$$

Thus, we are guaranteed to have $\left|f(x) - (-1)\right| < \varepsilon$ if $|x-1| > \dfrac{1}{\varepsilon}$, in particular, if $x > \dfrac{1}{\varepsilon} + 1$. Hence, setting $M \coloneqq \dfrac{1}{\varepsilon} + 1$ ensures that $|f(x) - (-1)| < \varepsilon$ whenever $x > M$.