Do low-complexity aperiodic SFTs exist?

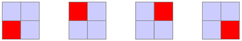
Problem posed by: Jarkko Kari
University of Turku

Presented by: Siamak Taati
American University of Beirut

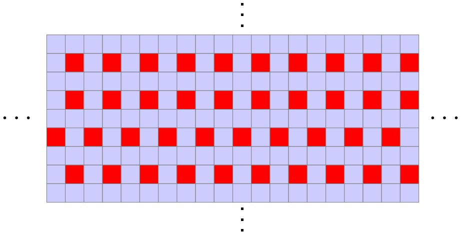
Expanding Dynamics — October 2020

A familiar question

Four patterns



A consistent configuration



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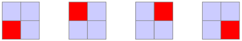

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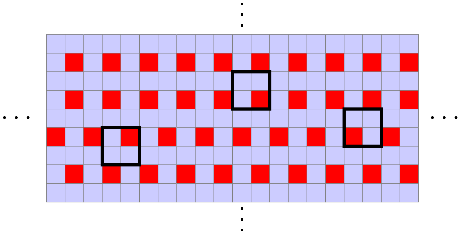

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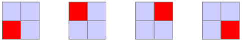

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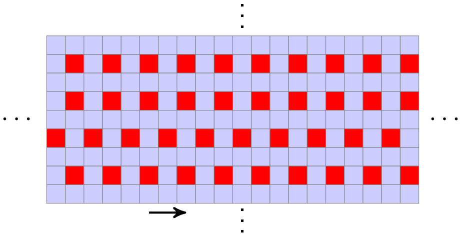

periodic in the horizontal direction

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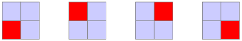

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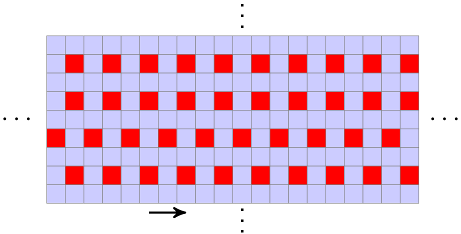

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Question: Do non-periodic consistent configurations exist?

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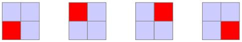

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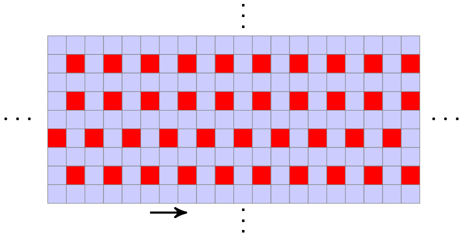

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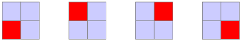
Question: Do non-periodic consistent configurations exist?

Nivat's conjecture (1997)

Every infinite configuration consistent with $\leq mn$ patterns with m-by-n rectangular shape is periodic in at least one direction.

A familiar question

Four patterns



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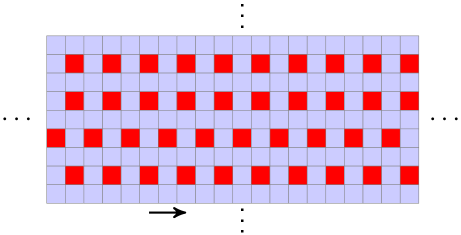

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- True in dimension $d = 1$ [Morse and Hedlund, 1938]

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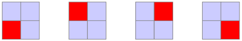

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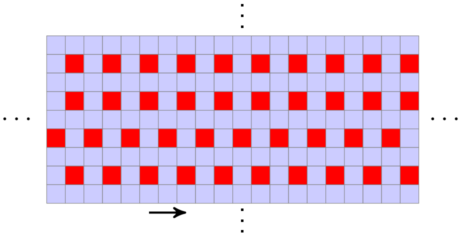

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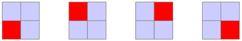

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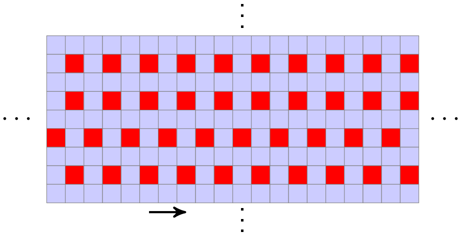

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- Not true for arbitrary shapes [Cassaigne, 1999]

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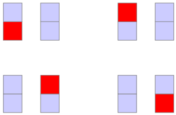

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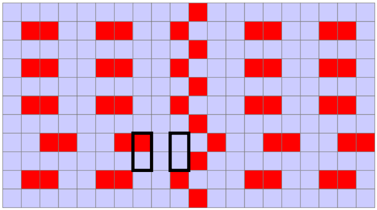

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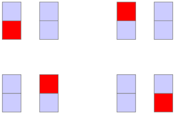

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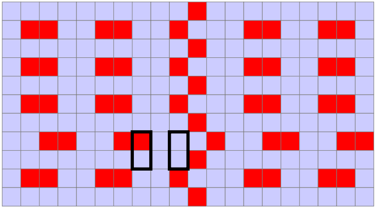

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- True in dimension $d = 1$ [Morse and Hedlund, 1938]
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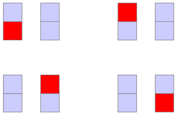
- Not true for arbitrary shapes [Cassaigne, 1999]
- . . . but perhaps for convex shapes? [Cassaigne, 1999]

A related question

Four patterns



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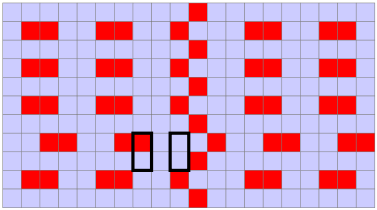

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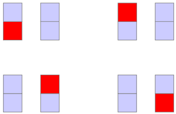

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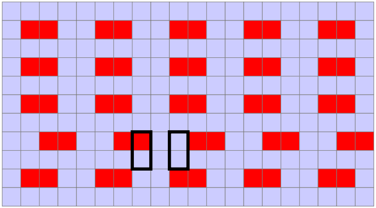

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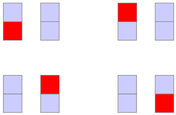

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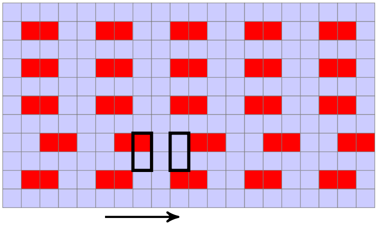

periodic in the horizontal direction

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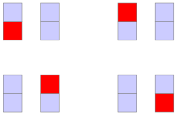

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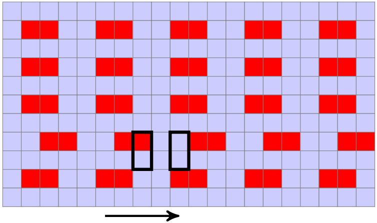


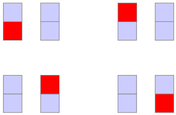
periodic in the horizontal direction

Jarkko's question

Does every consistent list of n patterns with the same n-cell shape admit a consistent *periodic* configuration?

A related question

Four patterns



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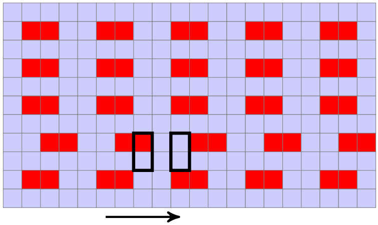

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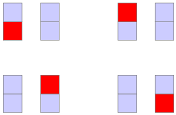
Does every consistent list of n patterns with the same n-cell shape admit a consistent *periodic* configuration?

- The shape is arbitrary!

A related question

Four patterns



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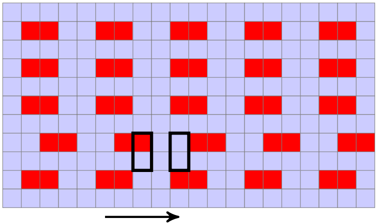


periodic in the horizontal direction

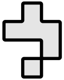
Jarkko's question

Does every consistent list of n patterns with the same n-cell shape admit a consistent *periodic* configuration?

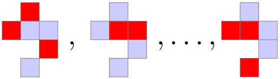
- The shape is arbitrary!
- Any number of dimensions!

Broader scenario

a shape

$D =$  $\Subset \mathbb{Z}^d$

a collection of patterns

$\mathcal{P} = \Big\{$  $\Big\} \subseteq \Sigma^D$

an alphabet

Broader scenario

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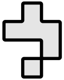
$D =$  $\in \mathbb{Z}^d$

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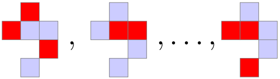
$\mathcal{P} = \Big\{$  $\Big\} \subseteq \Sigma^D$

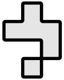
an alphabet

$X_{\mathcal{P}} = \{$all $\mathbb{Z}^d$-configurations consistent with $\mathcal{P}\}$

an SFT

Broader scenario

a shape

$D =$  $\Subset \mathbb{Z}^d$

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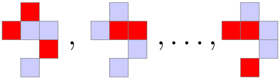
$\mathcal{P} = \{$  $\} \subseteq \Sigma^D$

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Q1

Is $X_{\mathcal{P}}$ non-empty?

Broader scenario

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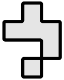
$D =$  $\Subset \mathbb{Z}^d$

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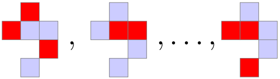
$\mathcal{P} = \{$  $\} \subseteq \Sigma^D$

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an SFT

Q1

Is $X_{\mathcal{P}}$ non-empty?

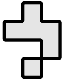
yes

Q1.1

Does $X_{\mathcal{P}}$ have periodic points?

Broader scenario

a shape

$D =$  $\Subset \mathbb{Z}^d$

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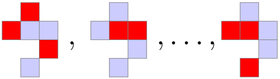
$\mathcal{P} = \Big\{$  $\Big\} \subseteq \Sigma^D$

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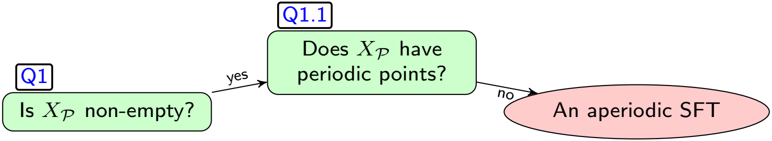

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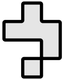
$D =$  $\in \mathbb{Z}^d$

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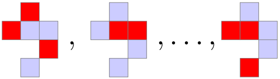
$\mathcal{P} = \Big\{$  $\Big\} \subseteq \Sigma^D$

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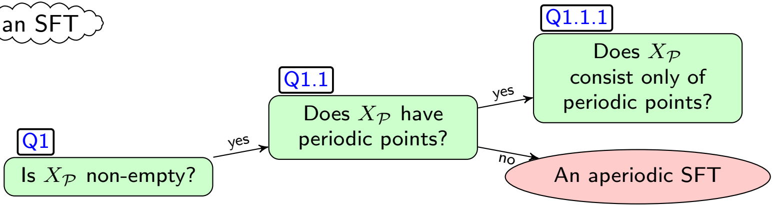

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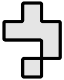
$D =$  $\in \mathbb{Z}^d$

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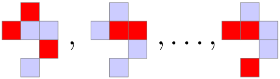
$\mathcal{P} = \Big\{$  $\Big\} \subseteq \Sigma^D$

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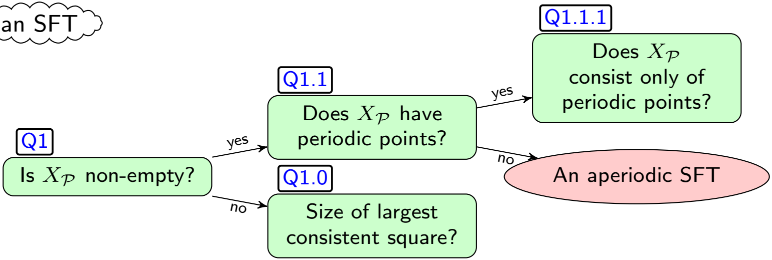

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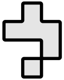
$D =$  $\Subset \mathbb{Z}^d$

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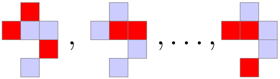
$\mathcal{P} = \Big\{$  $\Big\} \subseteq \Sigma^D$

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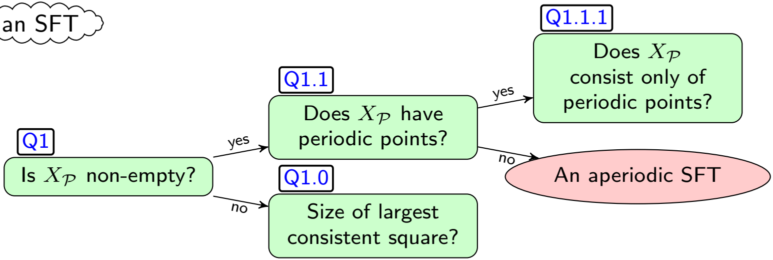
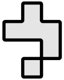


In dimension $d = 1$: All questions have simple answers.

- Q1, Q1.1.1, Q1.0 have simple algorithms. [e.g., via de Bruijn graph]
- The answer to Q1.1 is always positive.

Broader scenario

a shape

$D =$  $\Subset \mathbb{Z}^d$

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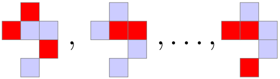
$\mathcal{P} = \Big\{$  $\Big\} \subseteq \Sigma^D$

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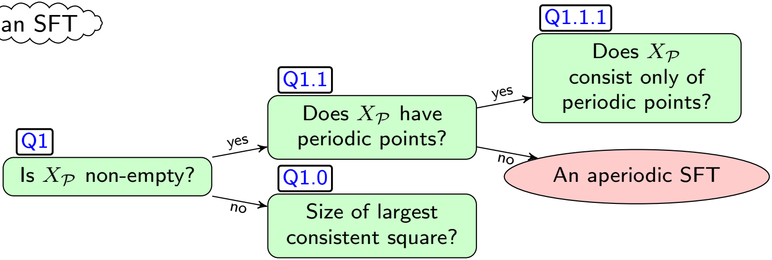
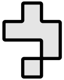


In dimensions $d \geq 2$: All questions are algorithmically unsolvable.

- Q1, Q1.1, Q1.1.1 are algorithmically undecidable.
- There is no computable bound for Q1.0.

Restricted variants

a shape

$D =$  $\Subset \mathbb{Z}^d$

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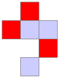
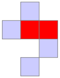
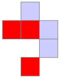
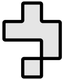
$\mathcal{P} = \Big\{$  ,  , $\ldots$,  $\Big\} \subseteq \Sigma^D$

Questions: What if $|\mathcal{P}| \leq |D|$?

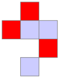
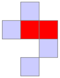
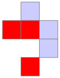
[the *low-complexity* case]

Restricted variants

a shape

$D =$  $\in \mathbb{Z}^d$

a collection of patterns

$\mathcal{P} = \{$  ,  , $\ldots$,  $\} \subseteq \Sigma^D$

Questions: What if $|\mathcal{P}| \leq |D|$? [the *low-complexity* case]

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Assuming $|\mathcal{P}| \leq |D|$ and D a rectangle (or convex),
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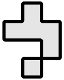
$D =$  $\subseteq \mathbb{Z}^d$

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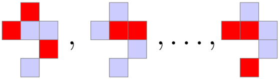
$\mathcal{P} = \{$  $\} \subseteq \Sigma^D$

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Restricted variants

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Dichotomy in dimension $d = 1$:

$\rightarrow$ Morse–Hedlund: if $|\mathcal{P}| \leq |D|$, then every consistent configuration is periodic.

$\rightarrow$ Sturmian configurations are non-periodic yet $|\mathcal{P}| = |D| + 1$ for every interval D.

(Recall: answer to Q1.1 is always positive in dimension 1.)

Restricted variants

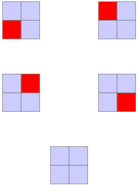
Questions: What is special about the threshold $|\mathcal{P}| \leq |D|$?

In dimension $d = 2$:

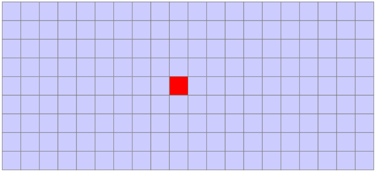
Nivat's conjecture if true would be optimal!

$\rightarrow$ Cassaigne (1999): A non-periodic example with $|\mathcal{P}| = |D| + 1$

Five patterns



A non-periodic consistent configuration



Restricted variants

Questions: What is special about the threshold $|\mathcal{P}| \leq |D|$?

In dimension $d = 2$:

Jarkko's conjecture if true would be *almost* optimal!

→ Kari (2020): For every $\varepsilon > 0$, question Q1 remains undecidable among instances where D is a rectangle and $|\mathcal{P}| \leq (1 + \varepsilon)\,|D|$.

→ If the answer to Jarkko's question is "Yes", then Q1 will be decidable for instances with $|\mathcal{P}| \leq |D|$.

[Simply run the two semi-algorithms in parallel!]

Restricted variants

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Assuming $|\mathcal{P}| \leq |D| + k$ and $\mathcal{P}$ consistent,
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Assuming $|\mathcal{P}| \leq |D| + k$ and $\mathcal{P}$ consistent,
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Note: Every aperiodic SFT gives a bound on k, above which the answer is negative. [e.g., negative if $k \geq 111$ based on Jeandel–Rao]

Low complexity terminology

For a configuration x:

$$L_D(x) := \{\text{all } D\text{-shaped patterns occurring in } x\}$$

Let us say x has low D-complexity if $|L_D(x)| \leq |D|$.

For a subshift X:

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Does there exist an aperiodic SFT that has low complexity w.r.t. some shape?

What is known?

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[Note: If D is not convex, then there are counter-examples.]

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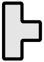
(K2) Connection with tilings with polyominoes ...

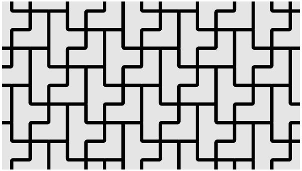
Tiling with polyominoes

A set of polyominoes

$\mathcal{T} = \{$  ,  $\}$

(allowed to be disconnected)

A tiling of $\mathbb{Z}^d$



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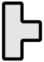
$\mathcal{T} = \{$  ,  $\}$

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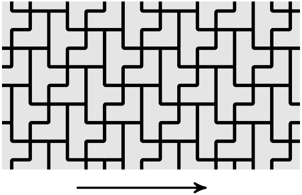

periodic in the horizontal direction

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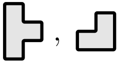
$\mathcal{T} = \{$  $\}$

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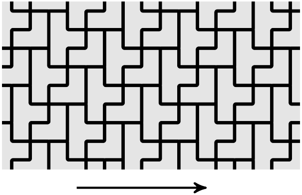

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Polyominoes can be encoded by allowed patterns and vice versa.

In particular, questions Q1, Q1.1, Q1.1.1, Q1.0 have equivalent forms in terms of tilings with polyominoes.

These questions are (by equivalence) undecidable/uncomputable.

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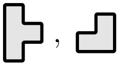
$\mathcal{T} = \{$  $\}$

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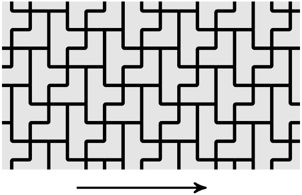

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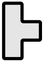
Question: What about restricted variants?

Tiling with a single polyomino

A polyomino

$F =$  $\in \mathbb{Z}^d$

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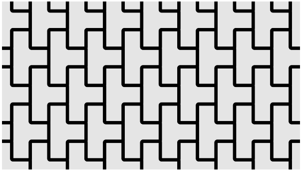

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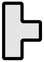
$F =$  $\Subset \mathbb{Z}^d$

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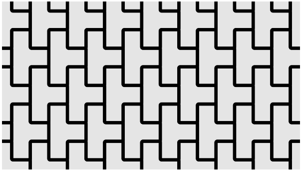
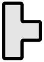
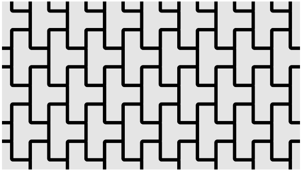
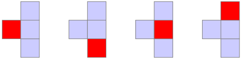
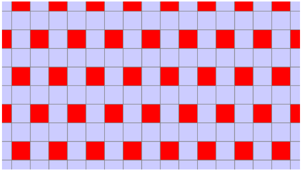


Periodic polyomino tiling conjecture

If a polyomino $F \Subset \mathbb{Z}^d$ can tile $\mathbb{Z}^d$, then it can also tile $\mathbb{Z}^d$ periodically.

[i.e., periodic in at least one direction]

Tiling with a single polyomino

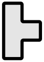
A polyomino	A tiling of $\mathbb{Z}^d$
$F =$  $\in \mathbb{Z}^d$	
Four patterns	**A consistent configuration**
	

There is a correspondence:

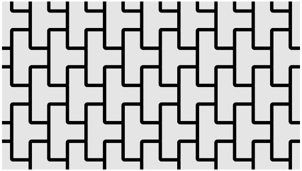
one polyomino $F \longleftrightarrow$ low complexity w.r.t. $D := -F$

Tiling with a single polyomino

A polyomino

$F =$  $\in \mathbb{Z}^d$

A tiling of $\mathbb{Z}^d$



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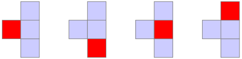

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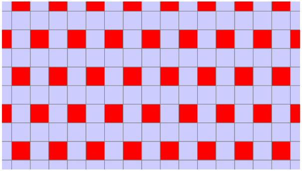


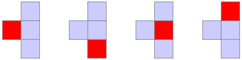
Hence, the periodic polyomino tiling question is a special case of Jarkko's question.

Tiling with a single polyomino

A polyomino

$F =$  $\in \mathbb{Z}^d$

Four patterns



What is known?

Tiling with a single polyomino

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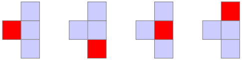

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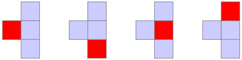

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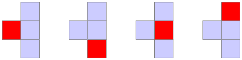

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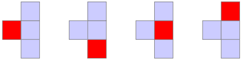

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The END.