Relative Gibbs measures and relative equilibrium measures

Joint work with
Sebastián Barbieri, Ricardo Gómez Aíza and Brian Marcus

Siamak Taati
Bernoulli Institute, University of Groningen

Workshop on Symbolic Dynamical Systems
Oaxaca — April 2019

Summary

DLR theorem [Dobrushin, 1968; Lanford and Ruelle, 1969]
Equilibrium measures $\equiv$ shift-invariant Gibbs measures
(under some conditions)

Example: Ising model (spontaneous magnetization)

Summary

DLR theorem [Dobrushin, 1968; Lanford and Ruelle, 1969]

Equilibrium measures $\equiv$ shift-invariant Gibbs measures
(under some conditions)

Example: Ising model (spontaneous magnetization)

Relative DLR theorem [Barbieri, Gómez Aíza, Marcus, T., 2018]

A similar equivalence for systems possibly in contact with
a random environment.

Example: Ising model on percolation clusters
(spontaneous magnetization in an alloy)

Summary

DLR theorem [Dobrushin, 1968; Lanford and Ruelle, 1969]
Equilibrium measures $\equiv$ shift-invariant Gibbs measures
(under some conditions)

Example: Ising model (spontaneous magnetization)

Relative DLR theorem [Barbieri, Gómez Aíza, Marcus, T., 2018]
A similar equivalence for systems possibly in contact with
a random environment.

Example: Ising model on percolation clusters
(spontaneous magnetization in an alloy)

Other new features:

$\longrightarrow$ More general lattice (any countable amenable group)

$\longrightarrow$ More general hard constraints in one direction

Summary

Some earlier works

- Relative DLR (different setting) [Seppäläinen, 1995]
- DLR on countable amenable groups (different setting) [Moulin Ollagnier and Pinchon, 1981; Tempelman, 1984]
- In the context of random dynamical systems (in 1d) [. . . ; Kifer and Liu, 2006; Kifer, 2008]

Summary

Some applications/corollaries

I. Equilibrium measures relative to a topological factor

[Extending a result of Allahbakhshi and Quas, 2013]

II. A local/global characterization of equilibrium measures

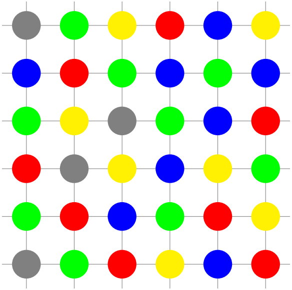
III. Relative version of Meyerovitch's theorem [Meyerovitch, 2013]

IV. Equilibrium measures on group shifts

[e.g., a sufficient condition for uniqueness of measure of max-entropy]

Example I (random colorings of $\mathbb{Z}^d$)

Let $X \subseteq \Sigma^{\mathbb{Z}^d}$ be the subshift consisting of all valid colorings of $\mathbb{Z}^d$ with a finite set of colors Σ, where $|\Sigma| > 2d$.



This is a strongly irreducible SFT.

Example I (random colorings of $\mathbb{Z}^d$)

Question

How does a "typical" configuration in X look like?

[typical $\rightsquigarrow$ as random as possible]

Answer 1 (global randomness)

A sample from a shift-invariant probability measure μ that maximizes entropy per site $h_\mu(X)$.

Answer 2 (local randomness)

A sample from a probability measure μ that is uniform Gibbs.

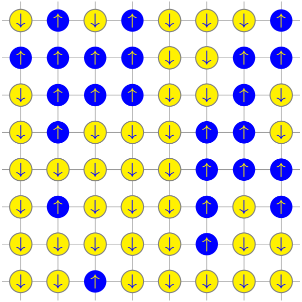
[...hence maximizing entropy locally]

According to the DLR theorem: [rediscovered by Burton and Steif, 1994]

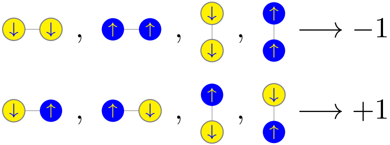
$\longrightarrow$ Among shift-invariant measures, local and global randomness are equivalent!

Example II (Ising model)

Let $X := \{\uparrow, \downarrow\}^{\mathbb{Z}^d}$. [upward/downward magnets at each site]



Interaction energies:



Example II (Ising model)

Question

How does a "typical" configuration in thermal equilibrium look like?

Answer 1 (global equilibrium)

A sample from a shift-invariant probability measure μ that maximizes pressure per site $\psi(\mu) := h_\mu(X) - \frac{1}{T}\mu(f_\Phi)$.

Answer 2 (local equilibrium)

A sample from a probability measure μ that is Gibbs for Φ.

[...hence maximizing pressure locally]

According to the DLR theorem:

$\longrightarrow$ Among shift-invariant measures, local and global equilibrium conditions are equivalent!

Classic DLR theorem

Theorem [Dobrushin, 1968; Lanford and Ruelle, 1969]

Let X be a $\mathbb{Z}^d$-subshift.

Let Φ and absolutely summable interaction on X.

(a) (Dobrushin) Assume that X is D-mixing.
Then, every shift-invariant Gibbs measure for Φ is an equilibrium measure for f_Φ.

(b) (Lanford–Ruelle) Assume that X is of finite type.
Then, every equilibrium measure for f_Φ is a (shift-invariant) Gibbs measure for Φ.

Classic DLR theorem

Theorem [Dobrushin, 1968; Lanford and Ruelle, 1969]

Let X be a $\mathbb{Z}^d$-subshift.
Let Φ and absolutely summable interaction on X.

(a) (Dobrushin) Assume that X is D-mixing.
Then, every shift-invariant Gibbs measure for Φ is an equilibrium measure for f_Φ.

(b) (Lanford–Ruelle) Assume that X is of finite type.
Then, every equilibrium measure for f_Φ is a (shift-invariant) Gibbs measure for Φ.

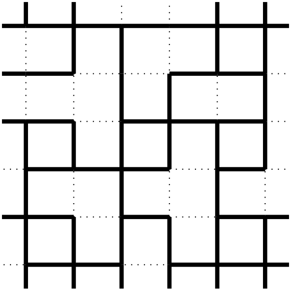
Remarks

1. $\mathbb{Z}^d$ can be replaced with any countable amenable group.
2. "D-mixing" is a relaxation of the uniform filling property.
3. The "finite type" condition can be replaced with weak topological Markov property (weak TMP).

Example III (random colorings of random graphs)

Let $\mathbb{G}$ be a finitely-generated amenable group and $(\mathbb{G}, \mathbb{E})$ the Cayley graph corresponding to a symmetric generator $S \notin 1_{\mathbb{G}}$. Let $(\mathbb{G}, \boldsymbol{\theta})$ be a $\mathbb{G}$-stationary random subgraph of $(\mathbb{G}, \mathbb{E})$.

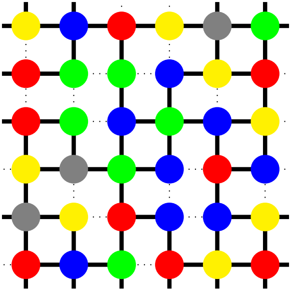
[e.g., bond percolation]



Example III (random colorings of random graphs)

Let $\mathbb{G}$ be a finitely-generated amenable group and $(\mathbb{G}, \mathbb{E})$ the Cayley graph corresponding to a symmetric generator $S \not\ni 1_{\mathbb{G}}$.
Let $(\mathbb{G}, \boldsymbol{\theta})$ be a $\mathbb{G}$-stationary random subgraph of $(\mathbb{G}, \mathbb{E})$.

[e.g., bond percolation]



Consider the valid Σ-colorings of $(\mathbb{G}, \boldsymbol{\theta})$, where $|\Sigma| > |S|$.

Example III (random colorings of random graphs)

Question

What is a "most random" Σ-coloring of $(\mathbb{G}, \boldsymbol{\theta})$?

Answer 1 (global randomness)

A random coloring $\boldsymbol{x}$ (defined on the same probability space as $\boldsymbol{\theta}$) that maximizes relative entropy per site $h(\boldsymbol{x} \,|\, \boldsymbol{\theta})$.

Answer 2 (local randomness)

A random coloring $\boldsymbol{x}$ (defined on the same probability space as $\boldsymbol{\theta}$) that is uniform Gibbs relative to $\boldsymbol{\theta}$.

[. . . hence maximizing entropy locally]

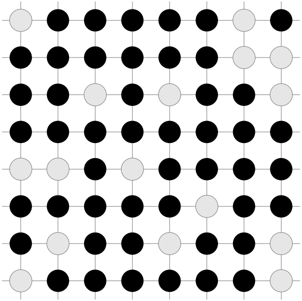
According to the relative DLR theorem:

$\longrightarrow$ Among shift-invariant measures with marginal ν,
local and global randomness are equivalent!

Example IV (Ising on percolation clusters)

Let $\Theta := \{\circ, \bullet\}^{\mathbb{Z}^d}$ and ν be the $\mathrm{Bernoulli}(p)$ measure on Θ.

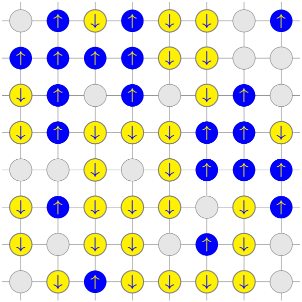
[. . . or any other shift-invariant measure]



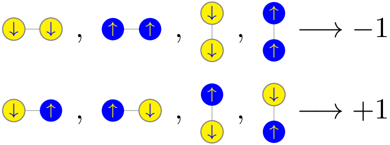
Example IV (Ising on percolation clusters)

Let $\Theta := \{\circ, \bullet\}^{\mathbb{Z}^d}$ and ν be the $\mathrm{Bernoulli}(p)$ measure on Θ.

[...or any other shift-invariant measure]



Consider the Ising model on the open clusters of the Bernoulli process.



Example IV (Ising on percolation clusters)

Question

What describes thermal equilibrium?

Answer 1 (global equilibrium)

A shift-invariant measure μ on Ω which has marginal ν on Θ and which maximizes relative pressure per site $\psi(\mu) := h_\mu(\Omega \mid \Theta) - \frac{1}{T}\mu(f_\Phi)$.

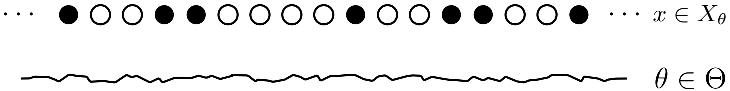
Answer 2 (local equilibrium)

A measure μ on Ω which has marginal ν on Θ and which is relative Gibbs for Φ. [...hence maximizing pressure locally]

According to the relative DLR theorem:

$\longrightarrow$ Among shift-invariant measures with marginal ν, local and global equilibrium conditions are equivalent!

Relative systems



Setting

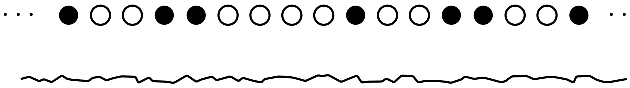
$\mathbb{G}$ the lattice: a countable amenable group

Θ the environment space: a measurable space on which $\mathbb{G}$ acts

X_θ configurations consistent with θ:
a non-empty closed subset of $\Sigma^{\mathbb{G}}$ for each $\theta \in \Theta$ s.t.

- (translation symmetry) $X_{g\theta} = gX_\theta$ for each $\theta \in \Theta$ and $g \in \mathbb{G}$,
- (measurability) $\Omega := \{(\theta, x) : \theta \in \Theta \text{ and } x \in X_\theta\}$ is measurable in $\Theta \times \Sigma^{\mathbb{G}}$.

Relative systems



$\cdots \; x \in X_\theta$

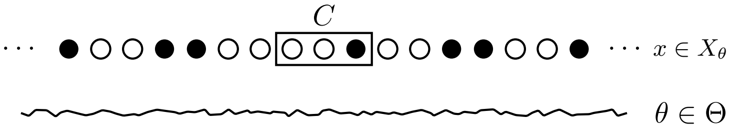
$\theta \in \Theta$

Setting

- $\mathbb{G}$ the lattice: a countable amenable group
- Θ the environment space: a measurable space on which $\mathbb{G}$ acts
- X_θ configurations consistent with θ:
  a non-empty closed subset of $\Sigma^{\mathbb{G}}$ for each $\theta \in \Theta$ s.t.
  - (translation symmetry) $X_{g\theta} = gX_\theta$ for each $\theta \in \Theta$ and $g \in \mathbb{G}$,
  - (measurability) $\Omega := \{(\theta, x) : \theta \in \Theta \text{ and } x \in X_\theta\}$ is measurable in $\Theta \times \Sigma^{\mathbb{G}}$.

Note: When $|\Theta| = 1$, we simply have a $\mathbb{G}$-subshift.

Relative systems



Interaction energies

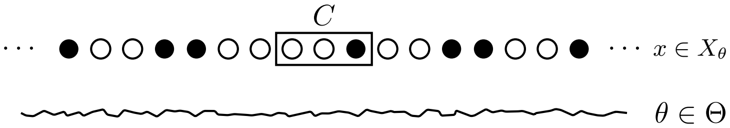
A family $\Phi := (\Phi_C)_{C \Subset \mathbb{G}}$ of measurable functions $\Phi : \Omega \to \mathbb{R}$ s.t.

- (relative locality) $\Phi_C(\theta, x)$ depends only on θ and x_C,
- (translation symmetry) $\Phi_{gC}(\theta, x) = \Phi_C(g^{-1}\theta, g^{-1}x)$.

We require absolute summability of the interactions:

$$\sum_{C \ni 1_{\mathbb{G}}} \|\Phi_C\| < \infty \, .$$

Relative systems



Interaction energies

A family $\Phi := (\Phi_C)_{C \Subset \mathbb{G}}$ of measurable functions $\Phi : \Omega \to \mathbb{R}$ s.t.

- (relative locality) $\Phi_C(\theta, x)$ depends only on θ and x_C,
- (translation symmetry) $\Phi_{gC}(\theta, x) = \Phi_C(g^{-1}\theta, g^{-1}x)$.

We require absolute summability of the interactions:

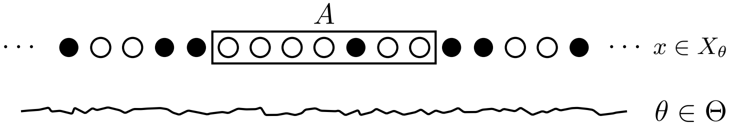
$$\sum_{C \ni 1_{\mathbb{G}}} \|\Phi_C\| < \infty \; .$$

Energy observable

Energy contribution of the site at the origin:

$$f_\Phi(\theta, x) := \sum_{C \ni 1_{\mathbb{G}}} \frac{1}{|C|} \Phi_C(\theta, x) \; .$$

Relative systems



Hamiltonian

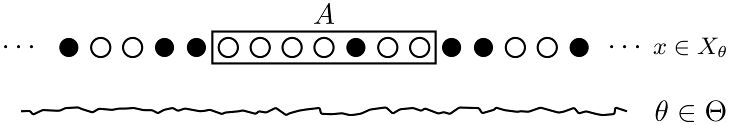
Energy content of a set $A \Subset \mathbb{G}$:

$$E_A(\theta, x) := \sum_{C \subseteq A} \Phi_C(\theta, x)$$

Energy of A and its interaction with the rest:

$$E_{A|A^{\mathsf{c}}}(\theta, x) := \sum_{\substack{C \Subset \mathbb{G} \\ C \cap A \neq \varnothing}} \Phi_C(\theta, x)$$

Relative systems



Hamiltonian

Energy content of a set $A \Subset \mathbb{G}$:

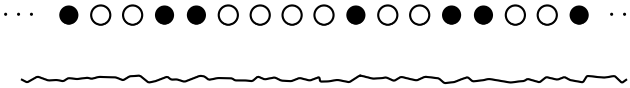
$$E_A(\theta, x) := \sum_{C \subseteq A} \Phi_C(\theta, x)$$

Energy of A and its interaction with the rest:

$$E_{A|A^{\mathsf{c}}}(\theta, x) := \sum_{\substack{C \Subset \mathbb{G} \\ C \cap A \neq \varnothing}} \Phi_C(\theta, x)$$

Note: Both E_A and $E_{A|A^{\mathsf{c}}}$ are relatively continuous.

Relative systems



$\cdots\ x \in X_\theta$

$\theta \in \Theta$

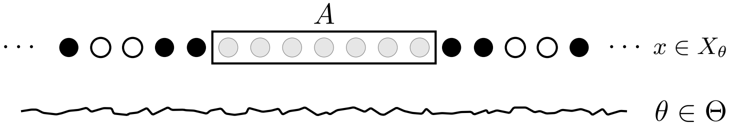
Relative equilibrium measures

Let ν be a $\mathbb{G}$-invariant measure on Θ.
An equilibrium measure for f_Φ relative to ν is a $\mathbb{G}$-invariant measure μ on Ω s.t.

(i) μ projects to ν,

(ii) μ maximizes relative pressure $\psi(\mu) := h_\mu(\Omega \,|\, \Theta) - \mu(f_\Phi)$ subject to (i).

Relative systems



Relative Gibbs measures

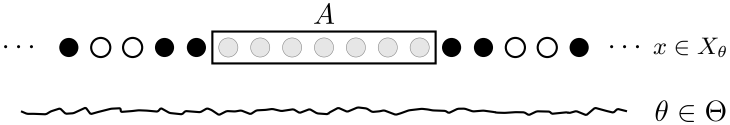
A relative Gibbs measure for Φ is a measure μ s.t.

- If $(\boldsymbol{\theta}, \boldsymbol{x}) \sim \mu$, then for every $A \Subset \mathbb{G}$,

$$\mathbb{P}(\boldsymbol{x}_A = u \,|\, \boldsymbol{\theta}, \boldsymbol{x}_{A^c}) = \begin{cases} \frac{1}{Z_{A|A^c}(\boldsymbol{\theta},\boldsymbol{x})} \mathrm{e}^{-E_{A|A^c}(\boldsymbol{\theta}, \boldsymbol{x}_{A^c} \vee u)} & \text{if } \boldsymbol{x}_{A^c} \vee u \in X_{\boldsymbol{\theta}}, \\ 0 & \text{otherwise,} \end{cases}$$

where $Z_{A|A^c}(\boldsymbol{\theta}, \boldsymbol{x})$ is the normalizing constant.

Relative systems



Relative Gibbs measures

A relative Gibbs measure for Φ is a measure μ s.t.

- If $(\boldsymbol{\theta}, \boldsymbol{x}) \sim \mu$, then for every $A \Subset \mathbb{G}$,

$$\mathbb{P}(\boldsymbol{x}_A = u \,|\, \boldsymbol{\theta}, \boldsymbol{x}_{A^\mathsf{c}}) = \begin{cases} \frac{1}{Z_{A|A^\mathsf{c}}(\boldsymbol{\theta},\boldsymbol{x})} \mathrm{e}^{-E_{A|A^\mathsf{c}}(\boldsymbol{\theta},\boldsymbol{x}_{A^\mathsf{c}} \vee u)} & \text{if } \boldsymbol{x}_{A^\mathsf{c}} \vee u \in X_{\boldsymbol{\theta}}, \\ 0 & \text{otherwise,} \end{cases}$$

where $Z_{A|A^\mathsf{c}}(\boldsymbol{\theta}, \boldsymbol{x})$ is the normalizing constant.

Recall: The Boltzmann distribution on a finite set is the unique distribution that maximizes pressure on that set.

Relative DLR theorem

Theorem [Barbieri, Gómez Aíza, Marcus, T., 2018]

Let $\Omega \subseteq \Theta \times \Sigma^{\mathbb{G}}$ be a relative system.
Let Φ be a absolutely summable relative interaction on Ω.
Let ν be a $\mathbb{G}$-invariant probability measure on Θ.

(a) (Relative D) Assume that Ω is D-mixing relative to ν.
Then, every $\mathbb{G}$-invariant relative Gibbs measure for Φ with marginal ν is an equilibrium measure for f_Φ relative to ν.

(b) (Relative LR) Assume that Ω is weak TMP relative to ν.
Assume further that Θ is standard Borel.
Then, every equilibrium measure for f_Φ relative to ν is a relative Gibbs measure for Φ (with marginal ν).

Remark

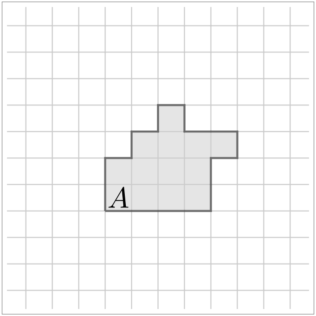
- If $|\Theta| = 1$, we recover classic DLR.

D-mixing

Let $X \subseteq \Sigma^{\mathbb{G}}$ be a non-empty closed set. [e.g., a subshift]

A mixing set for a set $A \subseteq \mathbb{G}$ in X is a set $B \supseteq A$ such that

- for every $x, y \in X$, there is a $z \in X$ with $z_A = x_A$ and $z_{B^c} = y_{B^c}$.

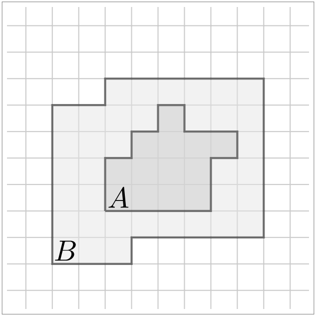


D-mixing

Let $X \subseteq \Sigma^{\mathbb{G}}$ be a non-empty closed set. [e.g., a subshift]

A mixing set for a set $A \subseteq \mathbb{G}$ in X is a set $B \supseteq A$ such that

- for every $x, y \in X$, there is a $z \in X$ with $z_A = x_A$ and $z_{B^c} = y_{B^c}$.

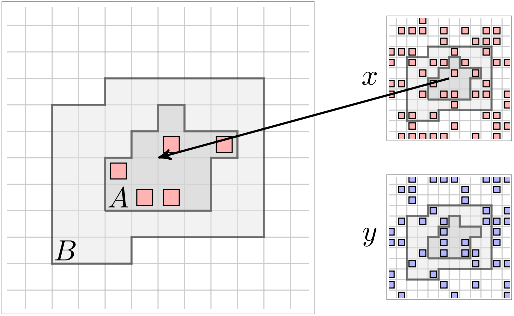


D-mixing

Let $X \subseteq \Sigma^{\mathbb{G}}$ be a non-empty closed set. [e.g., a subshift]

A mixing set for a set $A \subseteq \mathbb{G}$ in X is a set $B \supseteq A$ such that

- for every $x, y \in X$, there is a $z \in X$ with $z_A = x_A$ and $z_{B^c} = y_{B^c}$.

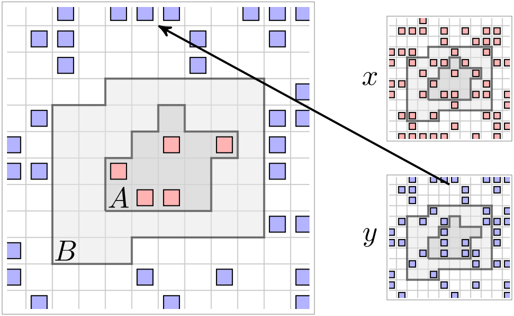


D-mixing

Let $X \subseteq \Sigma^{\mathbb{G}}$ be a non-empty closed set. [e.g., a subshift]

A mixing set for a set $A \subseteq \mathbb{G}$ in X is a set $B \supseteq A$ such that

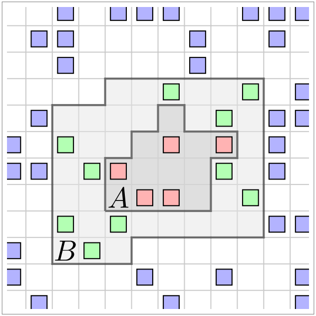
- for every $x, y \in X$, there is a $z \in X$ with $z_A = x_A$ and $z_{B^c} = y_{B^c}$.

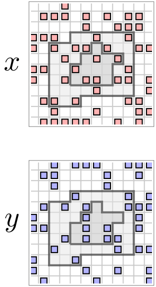


D-mixing

Let $X \subseteq \Sigma^{\mathbb{G}}$ be a non-empty closed set. [e.g., a subshift]
A mixing set for a set $A \subseteq \mathbb{G}$ in X is a set $B \supseteq A$ such that

- for every $x, y \in X$, there is a $z \in X$ with $z_A = x_A$ and $z_{B^c} = y_{B^c}$.



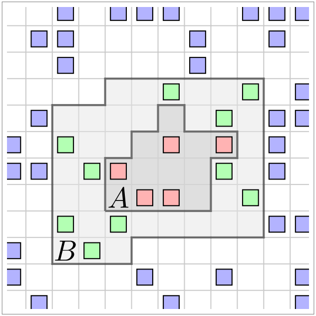


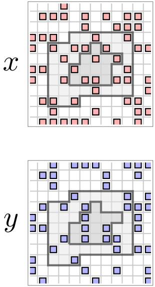
D-mixing

Let $X \subseteq \Sigma^{\mathbb{G}}$ be a non-empty closed set. [e.g., a subshift]

A mixing set for a set $A \subseteq \mathbb{G}$ in X is a set $B \supseteq A$ such that

- for every $x, y \in X$, there is a $z \in X$ with $z_A = x_A$ and $z_{B^c} = y_{B^c}$.





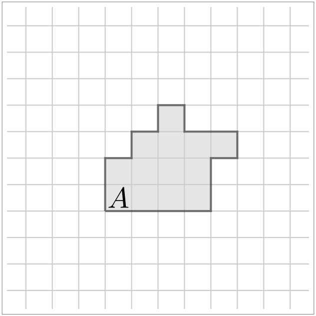
A $\mathbb{G}$-subshift X has is D-mixing if for some Følner sequence (F_n), each F_n has a mixing set $\overline{F}_n$ in X such that $|\overline{F}_n \setminus F_n| = o(n)$.

Weak topological Markov property (weak TMP)

Let $X \subseteq \Sigma^{\mathbb{G}}$ be a non-empty closed set. [e.g., a subshift]

A memory set for a set $A \subseteq \mathbb{G}$ in X is a set $B \supseteq A$ such that

- for every $x, y \in X$ with $x_{B \setminus A} = y_{B \setminus A}$, we have $x_B \vee y_{A^c} \in X$.

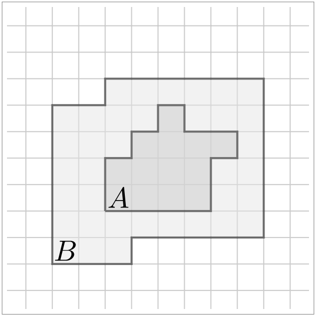


Weak topological Markov property (weak TMP)

Let $X \subseteq \Sigma^{\mathbb{G}}$ be a non-empty closed set. [e.g., a subshift]

A memory set for a set $A \subseteq \mathbb{G}$ in X is a set $B \supseteq A$ such that

- for every $x, y \in X$ with $x_{B \setminus A} = y_{B \setminus A}$, we have $x_B \vee y_{A^c} \in X$.

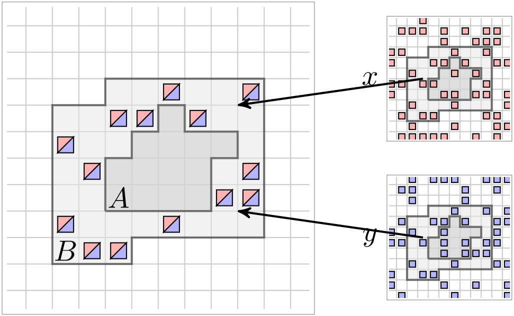


Weak topological Markov property (weak TMP)

Let $X \subseteq \Sigma^{\mathbb{G}}$ be a non-empty closed set. [e.g., a subshift]

A memory set for a set $A \subseteq \mathbb{G}$ in X is a set $B \supseteq A$ such that

- for every $x, y \in X$ with $x_{B \setminus A} = y_{B \setminus A}$, we have $x_B \vee y_{A^c} \in X$.

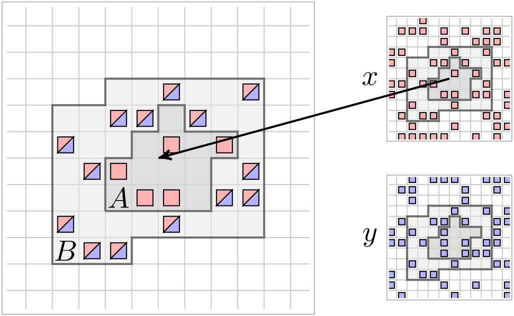


Weak topological Markov property (weak TMP)

Let $X \subseteq \Sigma^{\mathbb{G}}$ be a non-empty closed set. [e.g., a subshift]

A memory set for a set $A \subseteq \mathbb{G}$ in X is a set $B \supseteq A$ such that

- for every $x, y \in X$ with $x_{B \setminus A} = y_{B \setminus A}$, we have $x_B \vee y_{A^c} \in X$.

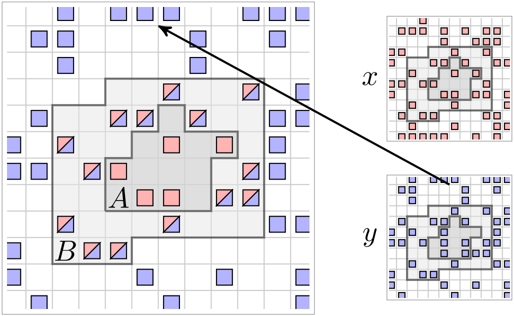


Weak topological Markov property (weak TMP)

Let $X \subseteq \Sigma^{\mathbb{G}}$ be a non-empty closed set. [e.g., a subshift]

A memory set for a set $A \subseteq \mathbb{G}$ in X is a set $B \supseteq A$ such that

- for every $x, y \in X$ with $x_{B \setminus A} = y_{B \setminus A}$, we have $x_B \vee y_{A^c} \in X$.

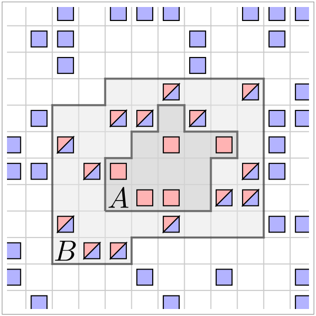


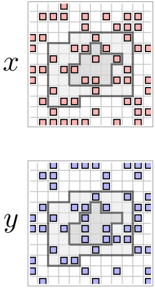
Weak topological Markov property (weak TMP)

Let $X \subseteq \Sigma^{\mathbb{G}}$ be a non-empty closed set. [e.g., a subshift]

A memory set for a set $A \subseteq \mathbb{G}$ in X is a set $B \supseteq A$ such that

- for every $x, y \in X$ with $x_{B \setminus A} = y_{B \setminus A}$, we have $x_B \vee y_{A^c} \in X$.





A $\mathbb{G}$-subshift X has the weak topological Markov property if

- every finite set $A \Subset \mathbb{G}$ has a finite memory set in X.

Relative DLR theorem

Some applications/corollaries

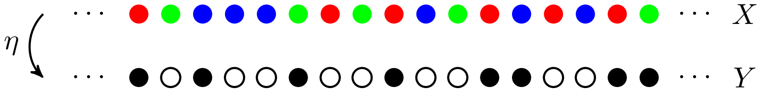
I. Equilibrium measures relative to a topological factor
[Extending a result of Allahbakhshi and Quas, 2013]

II. A local/global characterization of equilibrium measures

III. Relative version of Meyerovitch's theorem [Meyerovitch, 2013]

IV. Equilibrium measures on group shifts
[e.g., a sufficient condition for uniqueness of measure of max-entropy]

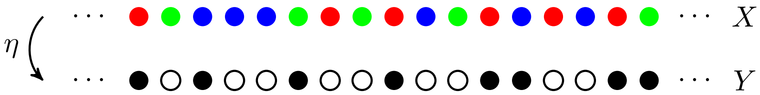
I. Equilibrium relative to a topological factor



Let $T : X \to X$ and $S : Y \to Y$ be continuous maps
on compact metric spaces X and Y.

Let $\eta : X \to Y$ be a topological factor map and
ν an S-invariant probability measure on Y.

I. Equilibrium relative to a topological factor

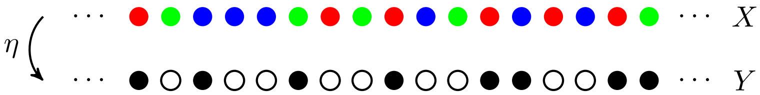


Let $T : X \to X$ and $S : Y \to Y$ be continuous maps on compact metric spaces X and Y.

Let $\eta : X \to Y$ be a topological factor map and ν an S-invariant probability measure on Y.

Ⓠ *Among the T-invariant measures μ on X with $\eta(\mu) = \nu$, which have maximum KS-entropy $h_\mu(X, T)$?*

I. Equilibrium relative to a topological factor

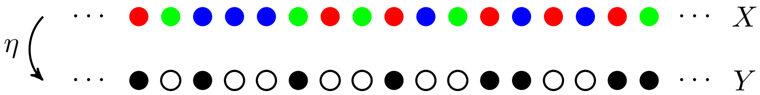


Let $T : X \to X$ and $S : Y \to Y$ be continuous maps
on compact metric spaces X and Y.
Let $\eta : X \to Y$ be a topological factor map and
ν an S-invariant probability measure on Y.

Q *Among the T-invariant measures μ on X with $\eta(\mu) = \nu$, which have maximum KS-entropy $h_\mu(X, T)$?*

- Ledrappier and Walters (1977) proved a "variational principle" for $h_\mu(X, T)$ relative to ν. [Also for pressure]

I. Equilibrium relative to a topological factor

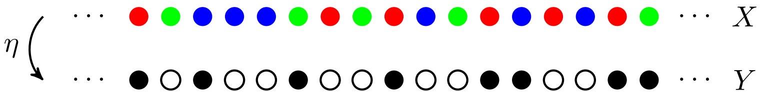


Let $T : X \to X$ and $S : Y \to Y$ be continuous maps
on compact metric spaces X and Y.
Let $\eta : X \to Y$ be a topological factor map and
ν an S-invariant probability measure on Y.

Q *Among the T-invariant measures μ on X with $\eta(\mu) = \nu$, which have maximum KS-entropy $h_\mu(X, T)$?*

- Ledrappier and Walters (1977) proved a "variational principle" for $h_\mu(X, T)$ relative to ν. [Also for pressure]
- When (X, T) is an SFT and (Y, T) is a sofic shift, Allahbakhshi and Quas (2013) showed that the maximizing measures have a uniform Gibbsian property relative to η.

I. Equilibrium relative to a topological factor



Corollary (of relative LR) [generalizes Allahbakhshi and Quas, 2013]

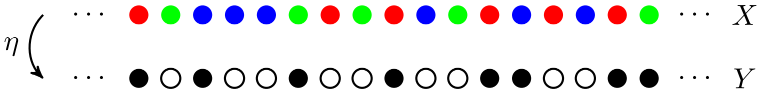
Let $\mathbb{G}$ be a countable amenable group.
Let X be a $\mathbb{G}$-subshift with weak TMP, and let $\eta : X \to Y$ be a topological factor map onto another $\mathbb{G}$-subshift Y.
Let Φ be an absolutely summable interaction on X and ν a $\mathbb{G}$-invariant measure on Y.
Let μ be a $\mathbb{G}$-invariant measure on X such that

(a) μ projects to ν,

(b) subject to (a), μ maximizes $h_\mu(X) - \mu(f_\Phi)$.

Then, μ has a Gibbsian property relative to η.

I. Equilibrium relative to a topological factor



The Gibbsian property

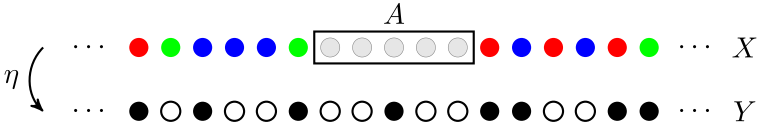
In the purely entropic case (i.e., $\Phi \equiv 0$):

- If $\boldsymbol{x} \sim \mu$, then for each $A \Subset \mathbb{G}$,

$$\mathbb{P}(\boldsymbol{x}_A \in \cdot \,|\, \boldsymbol{x}_{A^\mathsf{c}}, \eta(\boldsymbol{x}))$$

  is almost surely uniform over all patterns $u \in \Sigma^A$ that are consistent with $\boldsymbol{x}_A$ and $\eta(\boldsymbol{x})$.

I. Equilibrium relative to a topological factor



The Gibbsian property

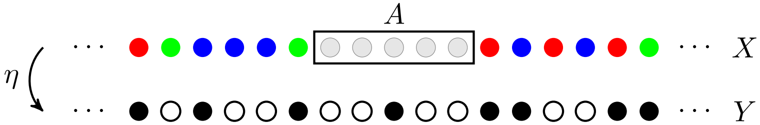
In the purely entropic case (i.e., $\Phi \equiv 0$):

- If $\boldsymbol{x} \sim \mu$, then for each $A \Subset \mathbb{G}$,

$$\mathbb{P}(\boldsymbol{x}_A \in \cdot \,|\, \boldsymbol{x}_{A^\mathsf{c}}, \eta(\boldsymbol{x}))$$

  is almost surely uniform over all patterns $u \in \Sigma^A$ that are consistent with $\boldsymbol{x}_A$ and $\eta(\boldsymbol{x})$.

I. Equilibrium relative to a topological factor



The Gibbsian property

In the purely entropic case (i.e., $\Phi \equiv 0$):

- If $\boldsymbol{x} \sim \mu$, then for each $A \Subset \mathbb{G}$,

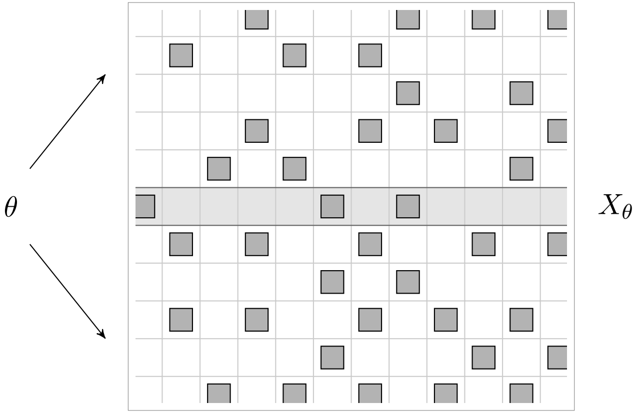
$$\mathbb{P}(\boldsymbol{x}_A \in \cdot \,|\, \boldsymbol{x}_{A^c}, \eta(\boldsymbol{x}))$$

  is almost surely uniform over all patterns $u \in \Sigma^A$ that are consistent with $\boldsymbol{x}_A$ and $\eta(\boldsymbol{x})$.

In the general case, the uniform distribution is replaced with the Boltzmann distribution.

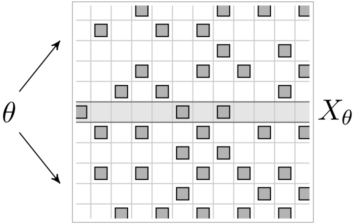
II. An in-between characterization of equilibrium measures

A $\mathbb{Z}^2$-subshift Y can be viewed as a relative $\mathbb{Z}$-system Ω_1.



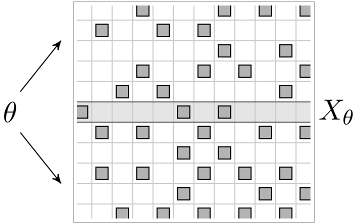
($\mathbb{Z}$ acts on Ω_1 by horizontal shift.)

II. An in-between characterization of equilibrium measures



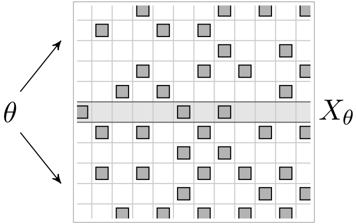
Let μ be a measure of maximal entropy on Y.

II. An in-between characterization of equilibrium measures



Let μ be a measure of maximal entropy on Y.
Suppose that Y has weak TMP.
$\Longrightarrow$ μ is uniform Gibbs on Y. [By Lanford–Ruelle]

II. An in-between characterization of equilibrium measures

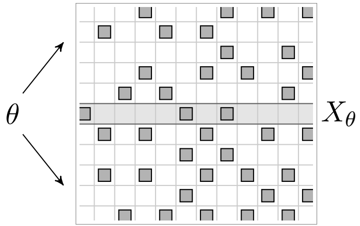


Let μ be a measure of maximal entropy on Y.
Suppose that Y has weak TMP.

$\Longrightarrow$ μ is uniform Gibbs on Y. [By Lanford–Ruelle]

$\Longrightarrow$ μ is uniform relative Gibbs on Ω_1.

II. An in-between characterization of equilibrium measures



Let μ be a measure of maximal entropy on Y.
Suppose that Y has weak TMP.

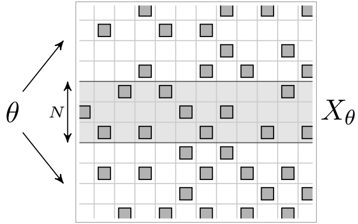
$\Longrightarrow$ μ is uniform Gibbs on Y. [By Lanford–Ruelle]

$\Longrightarrow$ μ is uniform relative Gibbs on Ω_1.

Suppose further that Ω satisfies relative D-mixing.

$\Longrightarrow$ μ maximizes $h_\mu(\Omega_1 \,|\, \Theta_1)$ among all the horizontally invariant measures with the same marginal on Θ_1. [By relative Dobrushin]

II. An in-between characterization of equilibrium measures



Let μ be a measure of maximal entropy on Y.
Suppose that Y has weak TMP.

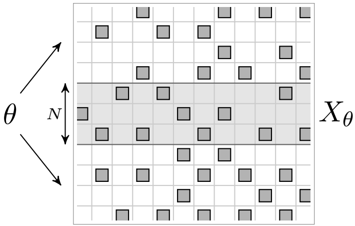
$\Longrightarrow$ μ is uniform Gibbs on Y. [By Lanford–Ruelle]

$\Longrightarrow$ μ is uniform relative Gibbs on Ω_N.

Suppose further that Ω satisfies relative D-mixing.

$\Longrightarrow$ μ maximizes $h_\mu(\Omega_N \,|\, \Theta_N)$ among all the horizontally invariant measures with the same marginal on Θ_N. [By relative Dobrushin]

II. An in-between characterization of equilibrium measures



Let μ be a measure of maximal entropy on Y.
Suppose that Y has weak TMP.

$\Longrightarrow$ μ is uniform Gibbs on Y. [By Lanford–Ruelle]

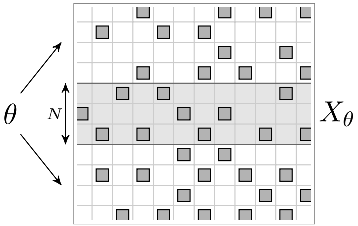
$\Longrightarrow$ μ is uniform relative Gibbs on Ω_N.

Suppose further that Ω satisfies relative D-mixing.

$\Longrightarrow$ μ maximizes $h_\mu(\Omega_N \,|\, \Theta_N)$ among all the horizontally invariant measures with the same marginal on Θ_N. [By relative Dobrushin]

Interpretation: μ is (conditionally) maximally random on every finite-width horizontal strip.

II. An in-between characterization of equilibrium measures



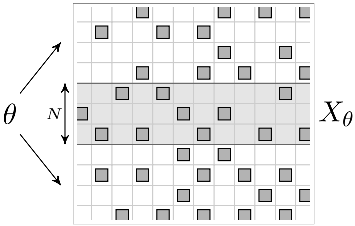
Corollary (of DLR and relative DLR)

Let Y be a $\mathbb{Z}^2$-subshift satisfying TSSM.
Let Φ be an absolutely summable interaction on Y
and μ a $\mathbb{Z}^2$-invariant probability measure on Y.
Then, the following are equivalent:

(i) μ is an equilibrium measure for f_Φ on Y.

(ii) For every $N \geq 1$,
μ is a relative equilibrium measure for f_Φ on Ω_N.

II. An in-between characterization of equilibrium measures



Remarks

1. More general setting:
   $\longrightarrow$ $\mathbb{Z}^2$ is replaced with a countable amenable group $\mathbb{G}$.
   $\longrightarrow$ Horizontal strips are replaced with $\mathbb{H}$-slices of $\mathbb{G}$ for a fixed subgroup $\mathbb{H} \subseteq \mathbb{G}$.
   (An $\mathbb{H}$-slice is a union of finitely manly cosets of $\mathbb{H}$.)
2. If $\mathbb{H}$ is the trivial subgroup $\{1_{\mathbb{G}}\}$, we recover DLR.

III. Meyerovitch's theorem and its relative version

Let X be an arbitrary subshift.

Two finite patterns $u, v \in L_A(X)$ are interchangeable in X if

- for every $x \in X$,

$$x_{A^c} \vee u \in X \qquad \text{if and only if} \qquad x_{A^c} \vee v \in X$$

Example 1: Golden mean shift

$$\cdots \; \bullet \circ \bullet \circ \circ \bullet \overset{u}{\boxed{\circ \bullet \circ}} \circ \bullet \circ \bullet \circ \circ \bullet \circ \; \cdots \qquad x_{A^c} \vee u$$

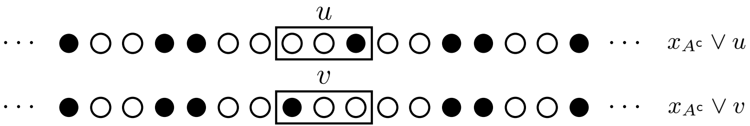
$$\cdots \; \bullet \circ \bullet \circ \circ \bullet \overset{v}{\boxed{\circ \circ \circ}} \circ \bullet \circ \bullet \circ \circ \bullet \circ \; \cdots \qquad x_{A^c} \vee v$$

Example 2: Even shift

$$\cdots \; \bullet \circ \circ \bullet \bullet \circ \circ \overset{u}{\boxed{\circ \circ \bullet}} \circ \circ \bullet \bullet \circ \circ \bullet \; \cdots \qquad x_{A^c} \vee u$$

$$\cdots \; \bullet \circ \circ \bullet \bullet \circ \circ \overset{v}{\boxed{\bullet \circ \circ}} \circ \circ \bullet \bullet \circ \circ \bullet \; \cdots \qquad x_{A^c} \vee v$$

III. Meyerovitch's theorem and its relative version



Meyerovitch's theorem [Meyerovitch, 2013]

Let X be an arbitrary $\mathbb{Z}^d$-subshift.
Let Φ be an absolutely summable interaction on X and μ an equilibrium measure for f_Φ.
Then, for every two interchangeable patterns $u, v \in L_A(X)$ and μ-almost every $x \in [u] \cup [v]$,

$$\frac{\mu([u] \mid \xi^{A^c})(x)}{\mathrm{e}^{-E_{A|A^c}(x_{A^c} \vee u)}} = \frac{\mu([v] \mid \xi^{A^c})(x)}{\mathrm{e}^{-E_{A|A^c}(x_{A^c} \vee v)}} .$$

III. Meyerovitch's theorem and its relative version

$$\cdots\ \bullet\circ\circ\bullet\bullet\circ\circ\overset{u}{\boxed{\circ\circ\bullet}}\circ\circ\bullet\bullet\circ\circ\bullet\ \cdots \qquad x_{A^c} \vee u$$

$$\cdots\ \bullet\circ\circ\bullet\bullet\circ\circ\overset{v}{\boxed{\bullet\circ\circ}}\circ\circ\bullet\bullet\circ\circ\bullet\ \cdots \qquad x_{A^c} \vee v$$

Meyerovitch's theorem

[Meyerovitch, 2013]

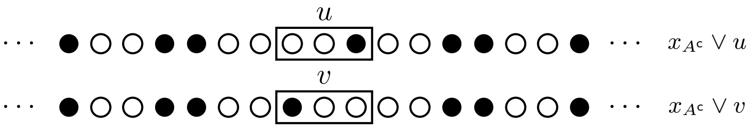
Let X be an arbitrary $\mathbb{Z}^d$-subshift.
Let Φ be an absolutely summable interaction on X and μ an equilibrium measure for f_Φ.
Then, for every two interchangeable patterns $u, v \in L_A(X)$ and μ-almost every $x \in [u] \cup [v]$,

$$\frac{\mu([u] \,|\, \xi^{A^c})(x)}{\mathrm{e}^{-E_{A|A^c}(x_{A^c} \vee u)}} = \frac{\mu([v] \,|\, \xi^{A^c})(x)}{\mathrm{e}^{-E_{A|A^c}(x_{A^c} \vee v)}} .$$

In the purely entropic case (i.e., $\Phi \equiv 0$), for $\boldsymbol{x} \sim \mu$ we get

$$\mathbb{P}(\boldsymbol{x}_A = u \,|\, \boldsymbol{x}_{A^c}) = \mathbb{P}(\boldsymbol{x}_A = v \,|\, \boldsymbol{x}_{A^c}) \qquad \text{almost surely.}$$

III. Meyerovitch's theorem and its relative version



Meyerovitch's theorem [Meyerovitch, 2013]

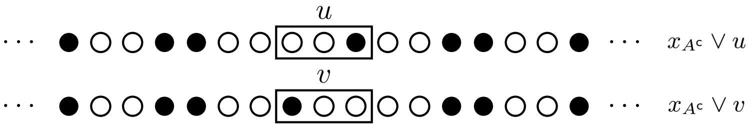
Let X be an arbitrary $\mathbb{Z}^d$-subshift.
Let Φ be an absolutely summable interaction on X and μ an equilibrium measure for f_Φ.
Then, for every two interchangeable patterns $u, v \in L_A(X)$ and μ-almost every $x \in [u] \cup [v]$,

$$\frac{\mu([u] \,|\, \xi^{A^c})(x)}{\mathrm{e}^{-E_{A|A^c}(x_{A^c} \vee u)}} = \frac{\mu([v] \,|\, \xi^{A^c})(x)}{\mathrm{e}^{-E_{A|A^c}(x_{A^c} \vee v)}} \,.$$

Remark

Meyerovitch's theorem generalizes the LR theorem!

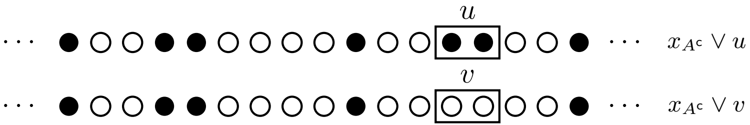
III. Meyerovitch's theorem and its relative version



Meyerovitch's theorem (entropic case) [Meyerovitch, 2013]

Let μ be a measure of maximal entropy on a $\mathbb{Z}^d$-subshift X.
Then, for every two interchangeable patterns $u, v \in L_A(X)$ we have $\mu([u]) = \mu([v])$.

III. Meyerovitch's theorem and its relative version



A nice extension of of Meyerovitch's theorem (entropic version):

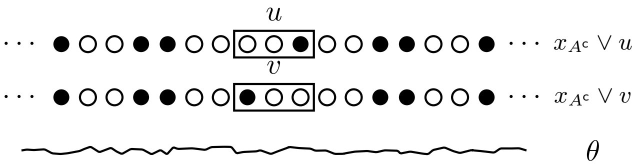
García–Pavlov Theorem [García-Ramos and Pavlov, 2018]

Let $\mathbb{G}$ be a countable amenable group.
Let μ be a measure of maximal entropy on a $\mathbb{G}$-subshift X.
Let $u, v \in L_A(X)$ be two finite patterns such that

- for every $x \in X$,
$$x_{A^c} \vee u \in X \qquad \Longrightarrow \qquad x_{A^c} \vee v \in X \ .$$

Then, $\mu([u]) \leq \mu([v])$.

III. Meyerovitch's theorem and its relative version



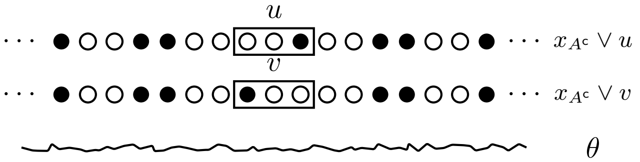
Let $\Omega \subseteq \Theta \times \Sigma^{\mathbb{G}}$ be a relative system.

Two finite patterns $u, v \in \Sigma^A$ are interchangeable in X_θ if

- for every $x \in X_\theta$,

$$x_{A^c} \vee u \in X_\theta \qquad \text{if and only if} \qquad x_{A^c} \vee v \in X_\theta$$

III. Meyerovitch's theorem and its relative version



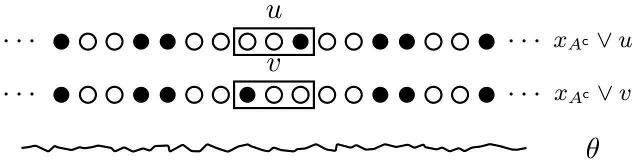
Relative version of Meyerovitch's theorem [BGMT, 2018]

Let $\mathbb{G}$ be countable amenable and $\Omega \subseteq \Theta \times \Sigma^{\mathbb{G}}$ a relative system.
Let Φ be an absolutely summable relative interaction on Ω
and ν a $\mathbb{G}$-invariant probability measure on Θ.
Let μ be an equilibrium measure for f_Φ relative to ν.
Then, for every two finite patterns $u, v \in \Sigma^A$,

$$\frac{\mu([u] \mid \xi^{A^c} \vee \mathscr{F}_\Theta)(\theta, x)}{\mathrm{e}^{-E_{A|A^c}(\theta, x_{A^c} \vee u)}} = \frac{\mu([v] \mid \xi^{A^c} \vee \mathscr{F}_\Theta)(\theta, x)}{\mathrm{e}^{-E_{A|A^c}(\theta, x_{A^c} \vee v)}}\ .$$

for μ-almost every $(\theta, x) \in [u] \cap [v]$ for which u and v are interchangeable in X_θ.

III. Meyerovitch's theorem and its relative version



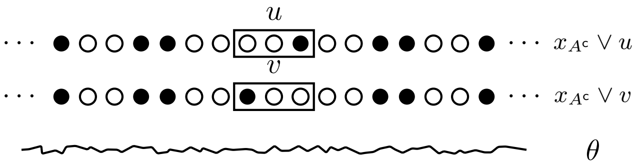
Relative version of Meyerovitch's theorem (entropic case) [BGMT, 2018]

Let $\mathbb{G}$ be countable amenable and $\Omega \subseteq \Theta \times \Sigma^{\mathbb{G}}$ a relative system.
Let ν be a $\mathbb{G}$-invariant probability measure on Θ.
Let μ be a measure on Ω which has marginal ν and which maximizes $h_\mu(\Omega \,|\, \Theta)$.
Then, for every two finite patterns $u, v \in \Sigma^A$,

$$\mathbb{P}(\boldsymbol{x}_A = u \,|\, \boldsymbol{\theta}, \boldsymbol{x}_{A^c}) = \mathbb{P}(\boldsymbol{x}_A = v \,|\, \boldsymbol{\theta}, \boldsymbol{x}_{A^c})$$

almost surely when $\boldsymbol{\theta} \in \Theta_{u,v}$.

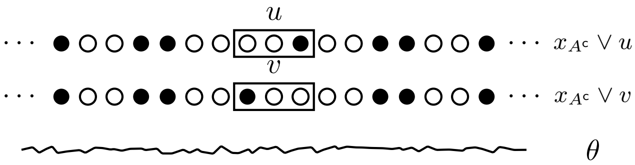
III. Meyerovitch's theorem and its relative version



Remarks

1. If $|\Theta| = 1$, we recover Meyerovitch's theorem.

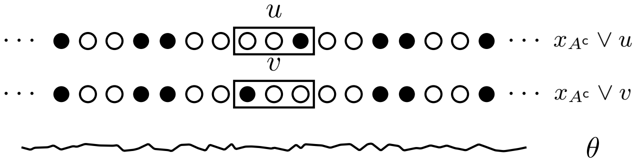
III. Meyerovitch's theorem and its relative version



Remarks

1. If $|\Theta| = 1$, we recover Meyerovitch's theorem.
2. The relative version of Meyerovitch's theorem generalizes the relative LR theorem.

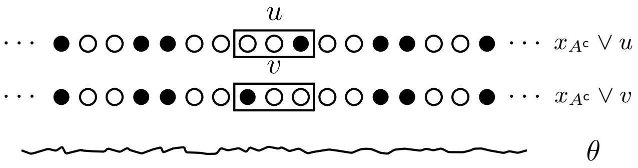
III. Meyerovitch's theorem and its relative version



Remarks

1. If $|\Theta| = 1$, we recover Meyerovitch's theorem.
2. The relative version of Meyerovitch's theorem generalizes the relative LR theorem.
3. The relative version of Meyerovitch's theorem follows from the relative LR theorem. [via a coding argument!]

III. Meyerovitch's theorem and its relative version



Remarks

1. If $|\Theta| = 1$, we recover Meyerovitch's theorem.
2. The relative version of Meyerovitch's theorem generalizes the relative LR theorem.
3. The relative version of Meyerovitch's theorem follows from the relative LR theorem. [via a coding argument!]

relative Meyerovitch on arbitrary relative systems $\equiv$ relative LR on relative systems having relative weak TMP

III. Meyerovitch's theorem and its relative version

$$\cdots \; \bullet \circ \circ \bullet \bullet \circ \circ \overset{u}{\boxed{\circ \circ \bullet}} \circ \circ \bullet \bullet \circ \circ \bullet \; \cdots \qquad x_{A^c} \vee u$$

$$\cdots \; \bullet \circ \circ \bullet \bullet \circ \circ \overset{v}{\boxed{\bullet \circ \circ}} \circ \circ \bullet \bullet \circ \circ \bullet \; \cdots \qquad x_{A^c} \vee v$$

Meyerovitch's theorem (entropic case) [Meyerovitch, 2013]

Let μ be a measure of maximal entropy on a $\mathbb{G}$-subshift X.
Then, for every two interchangeable patterns $u, v \in L_A(X)$

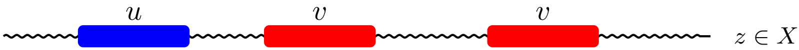
$$\mu([u] \,|\, \xi^{A^c}) = \mu([v] \,|\, \xi^{A^c}) \qquad \mu\text{-almost surely.}$$

Proof via relative LR ...

III. Meyerovitch's theorem and its relative version

Proof of Meyerovitch's theorem via relative LR (sketch).

Special case: <u>u and v are non-overlapping</u>

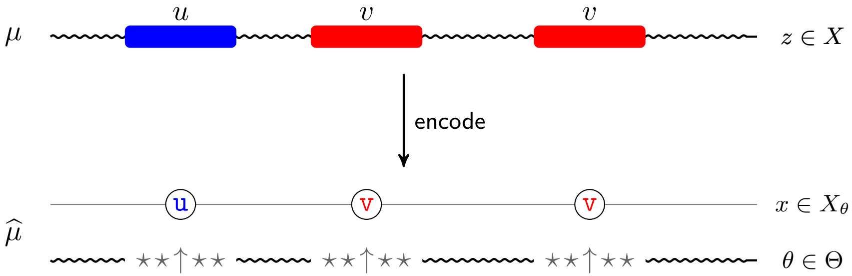


III. Meyerovitch's theorem and its relative version

Proof of Meyerovitch's theorem via relative LR (sketch).

Special case: u and v are non-overlapping

Encode X as a relative system Ω.

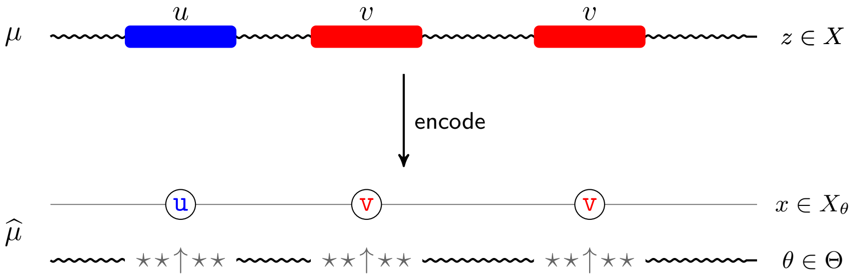


III. Meyerovitch's theorem and its relative version

Proof of Meyerovitch's theorem via relative LR (sketch).

Special case: u and v are non-overlapping

Encode X as a relative system Ω.
The new system has relative weak TMP.

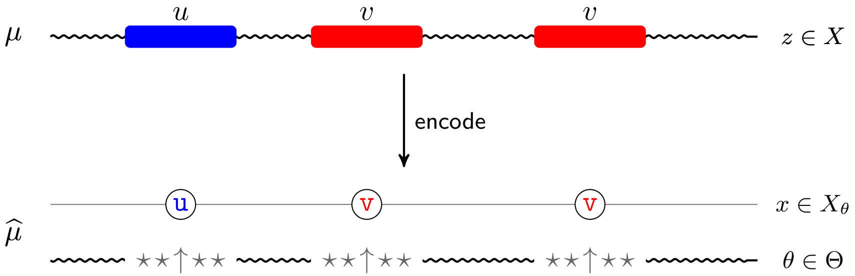


III. Meyerovitch's theorem and its relative version

Proof of Meyerovitch's theorem via relative LR (sketch).

Special case: u and v are non-overlapping

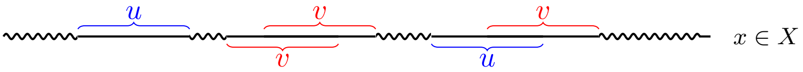
Encode X as a relative system Ω.
The new system has relative weak TMP.
Apply relative LR to the measure $\widehat{\mu}$ induced by μ on Ω.



III. Meyerovitch's theorem and its relative version

Proof of Meyerovitch's theorem via relative LR (sketch).

General case: u and v may overlap

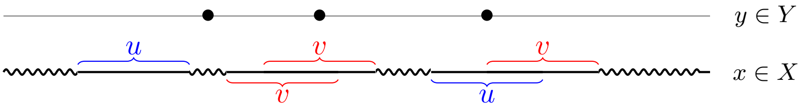


III. Meyerovitch's theorem and its relative version

Proof of Meyerovitch's theorem via relative LR (sketch).

General case: u and v may overlap

Let $Y \subseteq \{\circ, \bullet\}^{\mathbb{G}}$ be the hard-core subshift with shape $\mathrm{supp}(u)$.

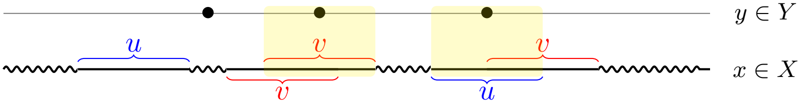


III. Meyerovitch's theorem and its relative version

Proof of Meyerovitch's theorem via relative LR (sketch).

General case: u and v may overlap

Let $Y \subseteq \{\circ, \bullet\}^{\mathbb{G}}$ be the hard-core subshift with shape $\mathrm{supp}(u)$.
The patterns $\begin{bmatrix} \bullet \\ u \end{bmatrix}$ and $\begin{bmatrix} \bullet \\ v \end{bmatrix}$ are
non-overlapping and interchangeable in $X \times Y$.

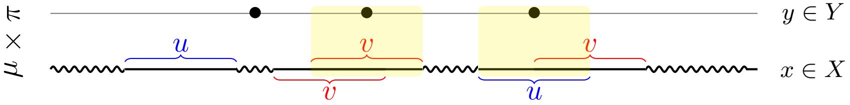


III. Meyerovitch's theorem and its relative version

Proof of Meyerovitch's theorem via relative LR (sketch).

General case: <u>u and v may overlap</u>

Let $Y \subseteq \{\circ, \bullet\}^{\mathbb{G}}$ be the hard-core subshift with shape $\mathrm{supp}(u)$.
The patterns $\begin{bmatrix}\bullet\\ u\end{bmatrix}$ and $\begin{bmatrix}\bullet\\ v\end{bmatrix}$ are
non-overlapping and interchangeable in $X \times Y$.
Apply the result of the non-overlapping case to $\mu \times \pi$ on $X \times Y$,
where π is the measure of maximal entropy on Y.



$\square$

IV: Equilibrium measures on group shifts

Let $\mathbb{G}$ be a countable group and $\mathbb{H}$ a finite group.
A group $\mathbb{G}$-shift is a subshift $\mathbb{X} \subseteq \mathbb{H}^{\mathbb{G}}$ which is also a subgroup of $\mathbb{H}^{\mathbb{G}}$.

Proposition [Kitchens and Schmidt, 1988]

Every group $\mathbb{Z}^d$-shift is of finite type.

Remark [see Salo, 2018]

If $\mathbb{G}$ is an arbitrary countable amenable group, then a group $\mathbb{G}$-shift may not be of finite type!

Proposition

Let $\mathbb{G}$ be a countable amenable group.
Then, every group $\mathbb{G}$-shift has weak TMP.

$\longrightarrow$ The extended LR theorem applies to group $\mathbb{G}$-shifts!

IV: Equilibrium measures on group shifts

A probability measure on μ on a compact metric group $\mathbb{X}$ is almost Haar if it is invariant under the action of the homoclinic subgroup $\Delta(\mathbb{X})$ of $\mathbb{X}$ by left-translations.

Proposition

Let $\mathbb{G}$ be a countable amenable group and $\mathbb{X}$ a group $\mathbb{G}$-shift. A probability measure on $\mathbb{X}$ is almost Haar if and only if it is uniform Gibbs.

Corollary (of extended LR)

Let $\mathbb{G}$ be a countable amenable group and $\mathbb{X}$ a group $\mathbb{G}$-shift. Suppose that the homoclinic subgroup $\Delta(\mathbb{X})$ is dense in $\mathbb{X}$. Then, the Haar measure on $\mathbb{X}$ is the unique measure of maximal entropy (w.r.t. the action of $\mathbb{G}$) on $\mathbb{X}$.

Thank you for your attention!