Metastability of the hard-core process on bipartite graphs

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METASTABILITY Workshop
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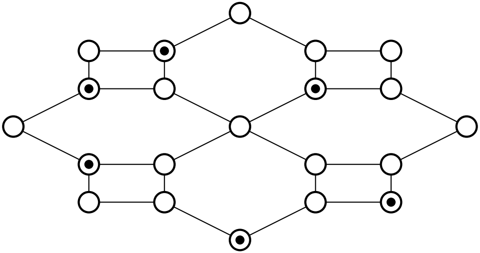
Hard-core gas process



Configurations

- Each site can carry at most one particle.
- Constraint: particles cannot site next to each other.

[Particles cannot overlap!]

Dynamics

- Birth attempt at site k (Poisson clock with rate λ_k)
- Death attempt at site k (Poisson clock with rate 1)
- All clocks are independent.

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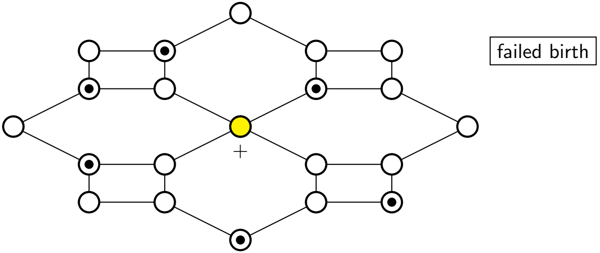

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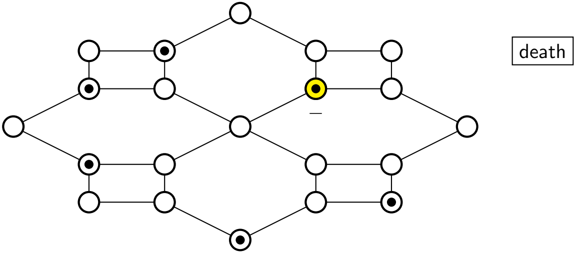

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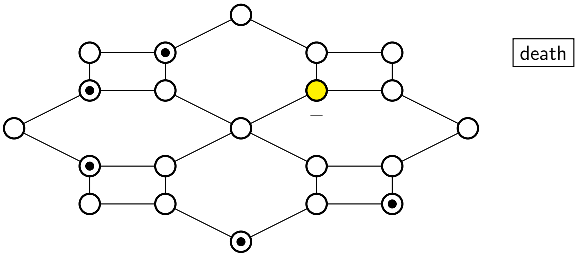

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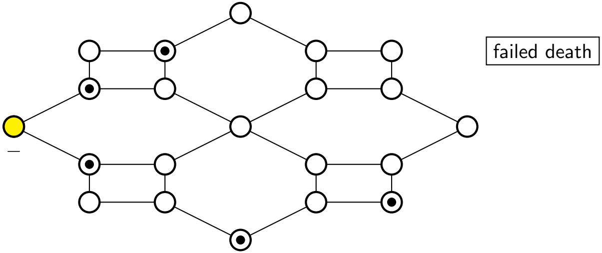

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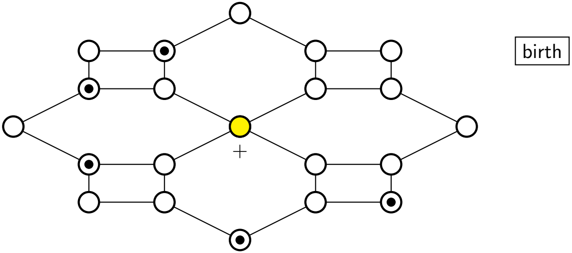

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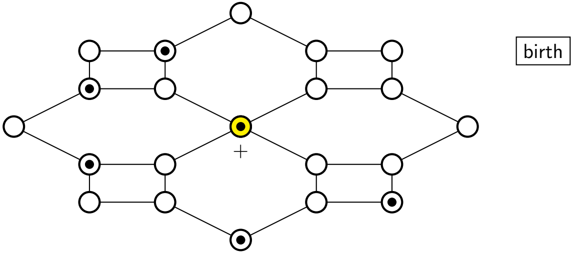

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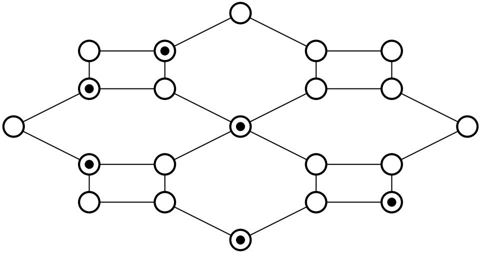

Reversible stationary distribution

[Boltzmann distribution]

$$\pi(x) = \frac{1}{Z} \prod_{\substack{k \text{ occupied} \\ \text{in } x}} \lambda_k$$

for each valid configuration x.
(Z is the appropriate normalizing constant.)

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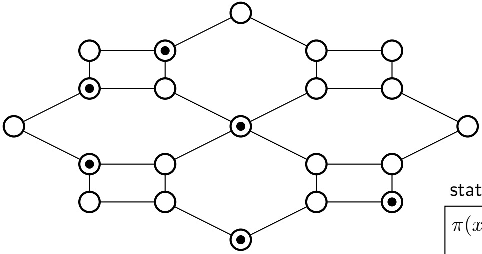

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Asymptotic regime

When the birth rates λ_k are very large:

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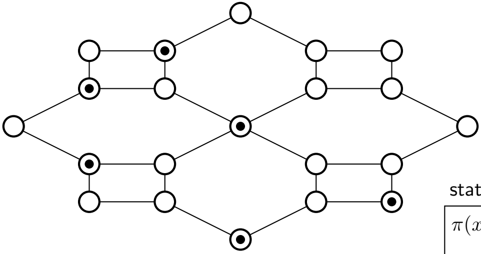

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When the birth rates λ_k are very large:

- The process tends to remain close to fully packed configurations, specially those that are "locally stable".

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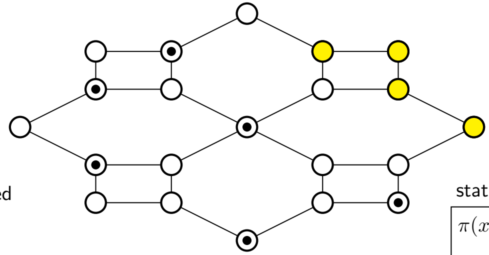

not fully packed

reversible
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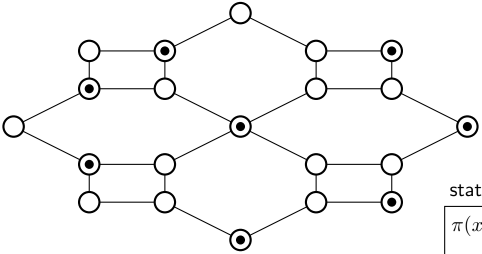
fully packed



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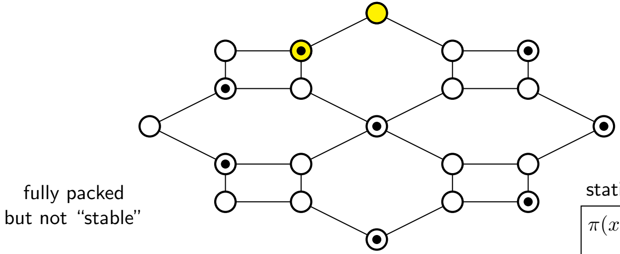

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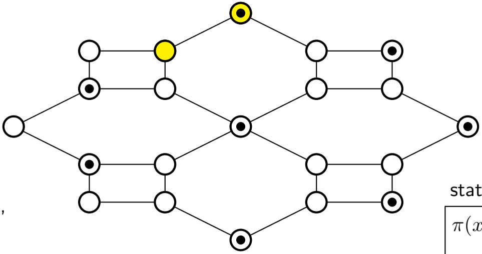

fully packed
but not "stable"

reversible
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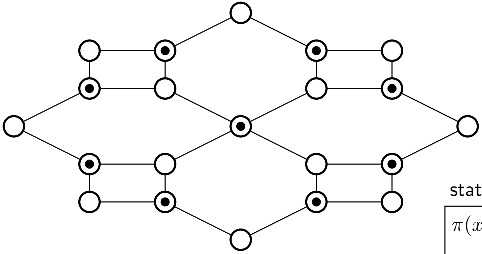
Hard-core gas process

fully packed and "stable"



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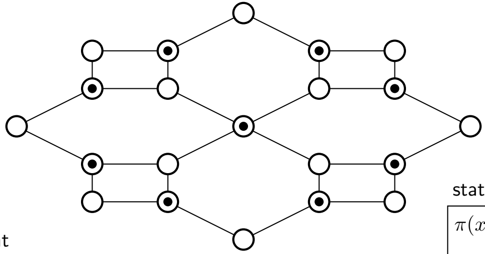

fully packed
and "stable"
but not efficient
(9)

reversible
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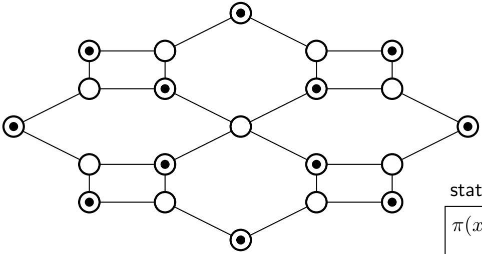
When the birth rates λ_k are very large:

- The process tends to remain close to fully packed configurations, specially those that are "locally stable".
- A typical stationary sample is efficiently fully packed!

Hard-core gas process



fully packed
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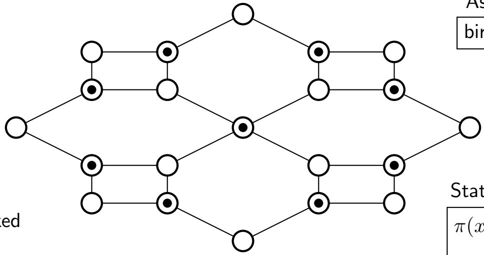

metastable
inefficiently packed
(9)

Asymptotic regime

birth rates λ_k large

Stationary distribution

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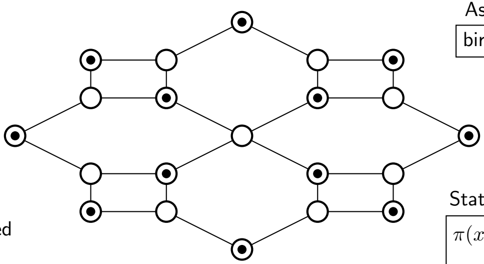
Metastability

- It takes a long time for the process to leave a "locally stable" but inefficiently packed configuration. [large exit time]
- Once a more efficient configuration is reached, it takes much longer to return. [small stationary probability]

Hard-core gas process



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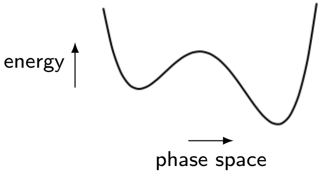
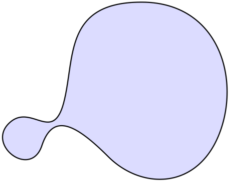
local minimum
in the energy landscape



bottleneck in
the phase space



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Hard-core gas on graphs

Motivation

- classic example from statistical mechanics [on the lattice]
  $\longrightarrow$ phase transition (solid-gas) with symmetry breaking
- wireless communication networks
  $\longrightarrow$ the graph represents the possibilities of interference
  $\longrightarrow$ metastability undermines the network performance
- includes the Widom-Rowlinson model

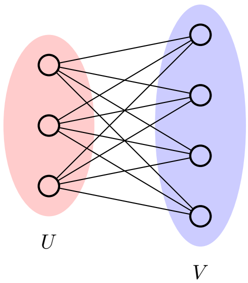
Related work

- Zocca, Borst, van Leeuwaarden and Nardi (2013–2015)
- Alessandro Zocca's PhD thesis (2015)
- Galvin and Tetali (2006), Randall (2008),
  — and Antonio Blanca (2012)

Hard-core gas process

An exagerated example

- complete bipartite graph
- birth rate λ at each site
- λ large
- $|U| < |V|$



Intuitive observations

- Exactly two fully packed configurations u and v

  $\longrightarrow$ Both u and v are "locally stable".

  $\longrightarrow$ v is "more efficient" than u.

- Metastable behaviour starting from u

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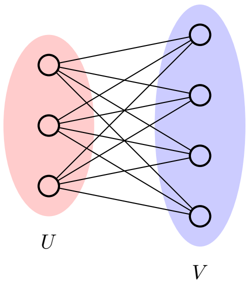

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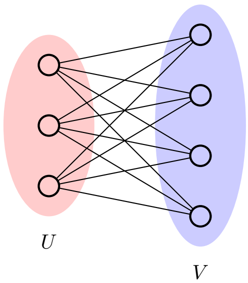

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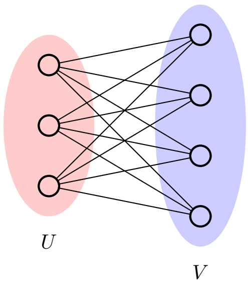

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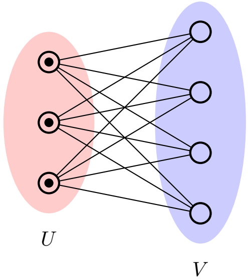

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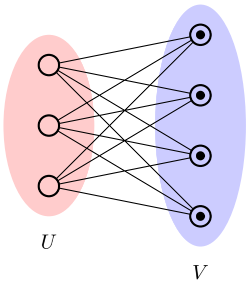

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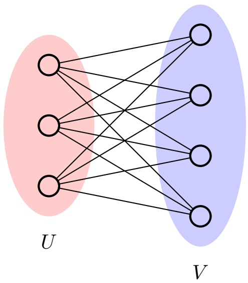

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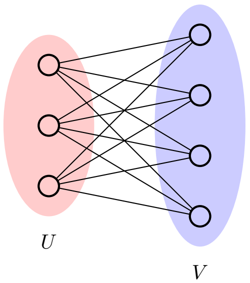

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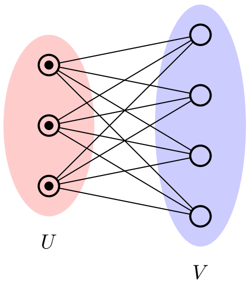

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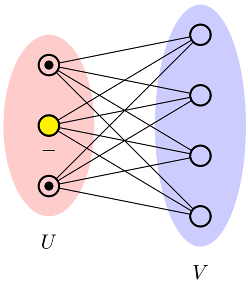

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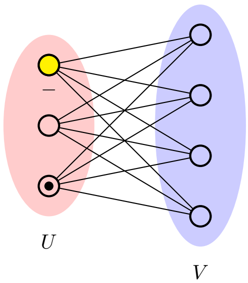

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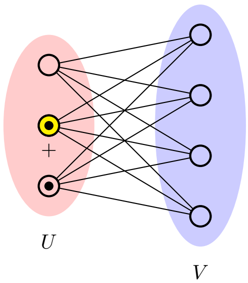

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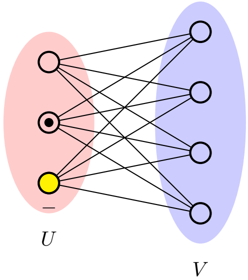

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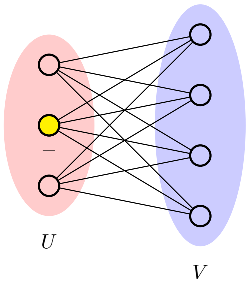

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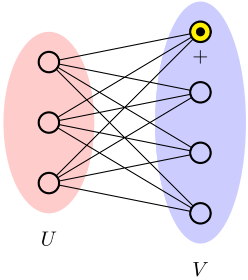

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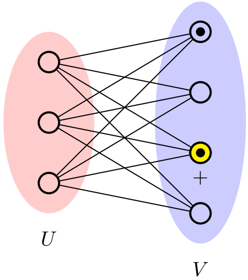

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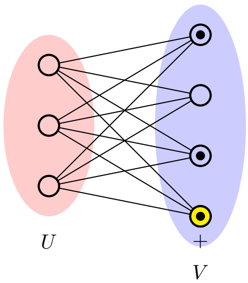

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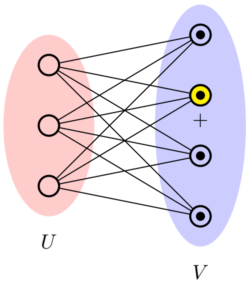

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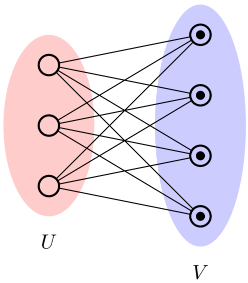


Intuitive observations

- Exactly two fully packed configurations u and v

  $\longrightarrow$ Both u and v are "locally stable".

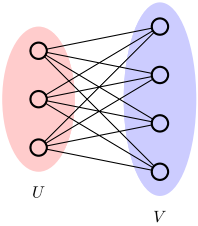
  $\longrightarrow$ v is "more efficient" than u.

- Metastable behaviour starting from u

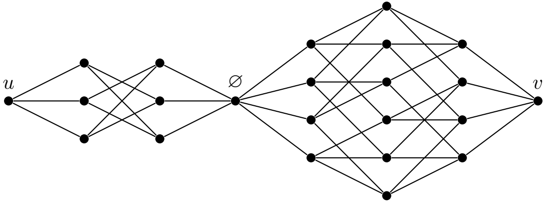
Question

How long does the transition from u to v take?

Hard-core processon a complete bipartite graph

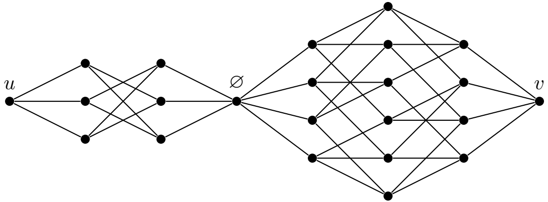


reversible Markov chain



Hard-core processon a complete bipartite graph

Reversible Markov chain

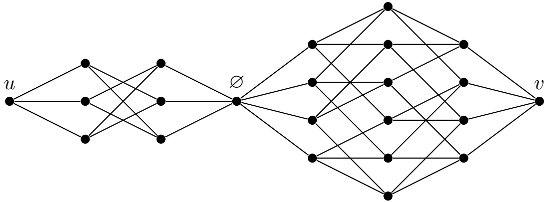


Let X_n $(n \geq 0)$ be the discrete-time Markov chain.
The first hitting time of v is

$$T_v := \inf\{n \geq 0 : X_n = v\} \ .$$

Hard-core processon a complete bipartite graph

Reversible Markov chain



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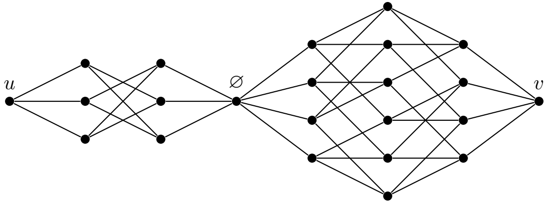
Hard-core processon a complete bipartite graph

Reversible Markov chain



Let X_n $(n \geq 0)$ be the discrete-time Markov chain.
The first hitting time of v is

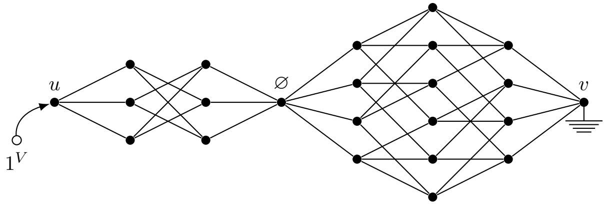
$$T_v := \inf\{n \geq 0 : X_n = v\} .$$

Question

What is the *expected transition time* $\mathbb{E}_u T_v$*?*

Hard-core processon a complete bipartite graph

As an electric network



Review: reversible Markov chain vs. electric network

Fundamental connection I

For every state x,

$$\mathbb{P}_x(T_A < T_B) = \text{voltage}(x)$$

if a 1^V battery is connected between A and B.

Fundamental connection II

For every state x,

$$G_{T_B}(a,x) = \overbrace{\mathcal{R}(a \leftrightarrow B)}^{\text{effective resistance}} \pi(x)\, \mathbb{P}_x(T_a < T_B)$$

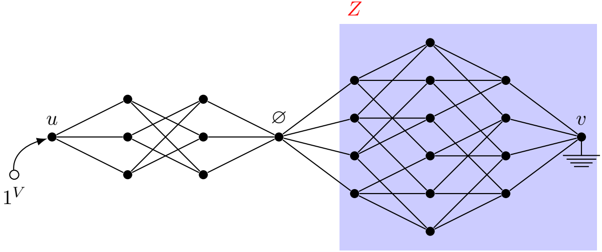
where $G_{T_B}(a,x) := \mathbb{E}_a[\#\text{ of visits to } x \text{ before } T_B]$.

Corollary

$$\mathbb{E}_a T_B = \mathcal{R}(a \leftrightarrow B) \sum_x \pi(x)\, \mathbb{P}_x(T_a < T_B)$$

Hard-core processon a complete bipartite graph

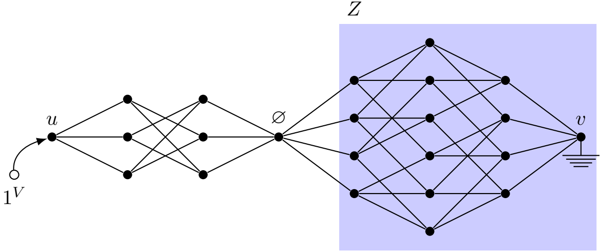
Expected transition time



$$\mathbb{E}_u T_v \approx \mathbb{E}_u T_Z$$
$$= \mathcal{R}(u \leftrightarrow Z) \sum_x \pi(x)\, \mathbb{P}_x(T_u < T_Z)$$
$$= \pi(u)\mathcal{R}(u \leftrightarrow Z) \sum_x \frac{\pi(x)}{\pi(u)}\, \mathbb{P}_x(T_u < T_Z)$$
$$\approx \pi(u)\mathcal{R}(u \leftrightarrow Z)$$

Hard-core processon a complete bipartite graph

Expected transition time



$$
\begin{aligned}
\mathbb{E}_u T_v &\approx \mathbb{E}_u T_Z \\
&= \mathcal{R}(u \leftrightarrow Z) \sum_x \pi(x)\, \mathbb{P}_x(T_u < T_Z) \\
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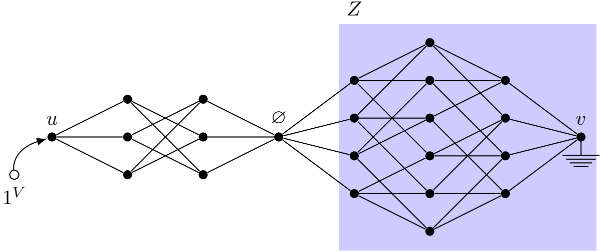
\end{aligned}
$$

Hard-core processon a complete bipartite graph

Expected transition time



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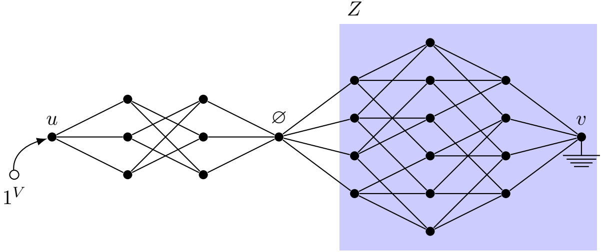
Hard-core processon a complete bipartite graph

Expected transition time



$$\mathbb{E}_u T_v \approx \mathbb{E}_u T_Z$$

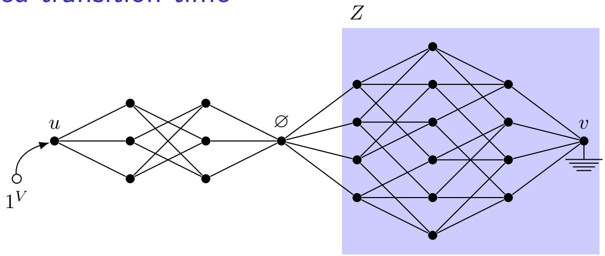
$$= \mathcal{R}(u \leftrightarrow Z) \sum_x \pi(x)\, \mathbb{P}_x(T_u < T_Z)$$

$$= \pi(u)\mathcal{R}(u \leftrightarrow Z) \left[1 + \sum_{x \neq u} \frac{\pi(x)}{\pi(u)}\, \mathbb{P}_x(T_u < T_Z) \right]$$

$$\approx \pi(u)\mathcal{R}(u \leftrightarrow Z)$$

Hard-core processon a complete bipartite graph

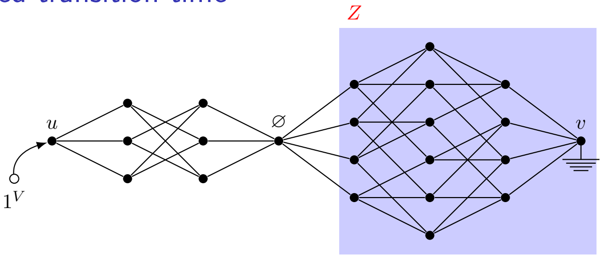
Expected transition time



$$
\begin{aligned}
\mathbb{E}_u T_v \approx\ & \mathbb{E}_u T_Z \\
&= \mathcal{R}(u \leftrightarrow Z) \sum_x \pi(x)\, \mathbb{P}_x(T_u < T_Z) \\
&= \pi(u)\mathcal{R}(u \leftrightarrow Z) \Big[1 + \sum_{x \neq u} \frac{\pi(x)}{\pi(u)}\, \mathbb{P}_x(T_u < T_Z) \Big] \\
&\approx \pi(u)\mathcal{R}(u \leftrightarrow Z)
\end{aligned}
$$

Hard-core processon a complete bipartite graph

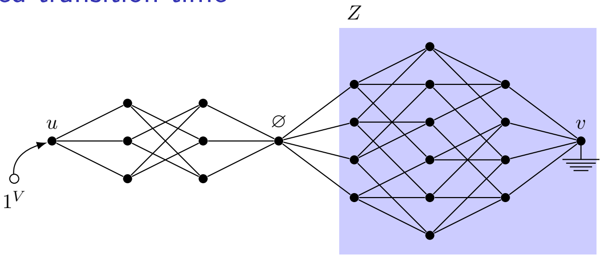
Expected transition time



$$\mathbb{E}_u T_v \approx \pi(u)\mathcal{R}(u \leftrightarrow Z) \approx \pi(u)\mathcal{R}(u \leftrightarrow v)$$

Hard-core processon a complete bipartite graph

Expected transition time

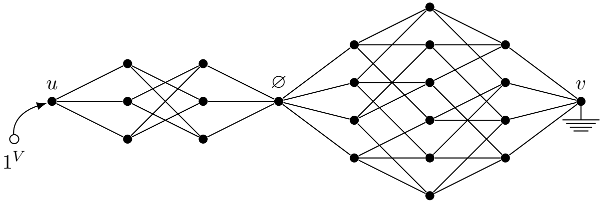


$$\mathbb{E}_u T_v \approx \pi(u)\mathcal{R}(u \leftrightarrow Z) \approx \pi(u)\mathcal{R}(u \leftrightarrow v)$$

It remains to estimate $\mathcal{R}(u \leftrightarrow v)$.

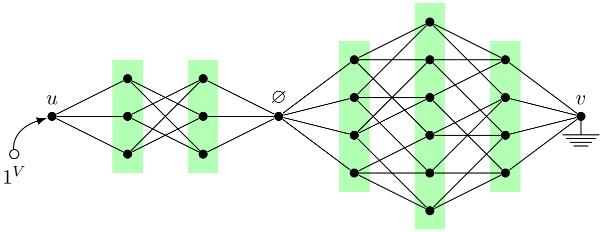
Hard-core processon a complete bipartite graph

Estimating the effective resistance



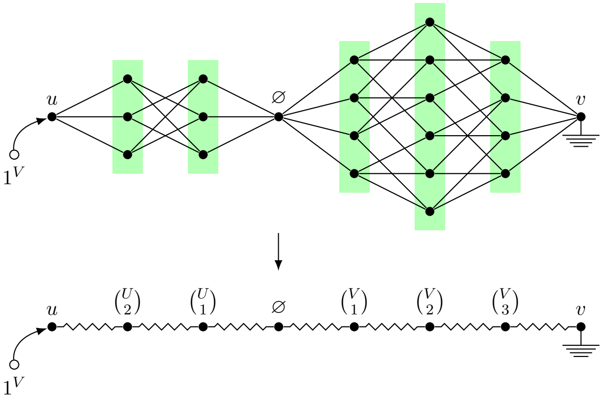
Hard-core processon a complete bipartite graph

Estimating the effective resistance



Hard-core processon a complete bipartite graph

Estimating the effective resistance



Hard-core processon a complete bipartite graph

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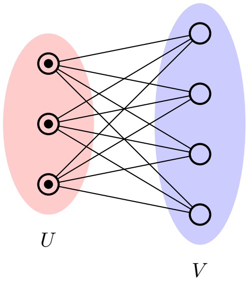
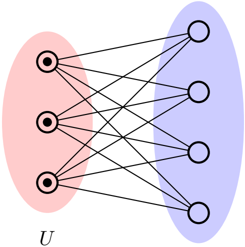


Proposition (Discrete time)

$$\mathbb{E}_u T_v = \frac{1}{|U|}\lambda^{|U|-1}[1 + o(1)] \qquad \textit{as } \lambda \to \infty.$$

Hard-core processon a complete bipartite graph

Expected transition time



Proposition (Continuous time)

$$\mathbb{E}_u T_v = \frac{\gamma}{|U|}\lambda^{|U|-1}[1+o(1)] \qquad \textit{as } \lambda \to \infty.$$

$\gamma := (|U|+|V|)(1+\lambda)$ *is the rate of Poisson clock*

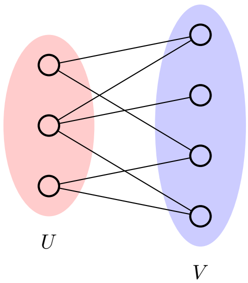
Hard-core process on a bipartite graph

A more general setting

- an arbitrary bipartite graph
- Birth rates

$$\begin{array}{ll} \lambda & \text{on } U \\ \bar{\lambda} & \text{on } V \end{array}$$

- $\lambda, \bar{\lambda}$ large



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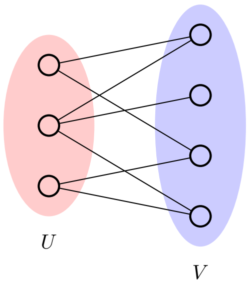

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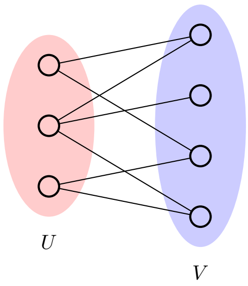

Hard-core process on a bipartite graph

A more general setting

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- Birth rates (with $0 < \alpha < 1$)

$$\begin{array}{ll} \lambda & \text{on } U \\ \bar{\lambda} = \lambda^{1+\alpha+o(1)} & \text{on } V \end{array}$$

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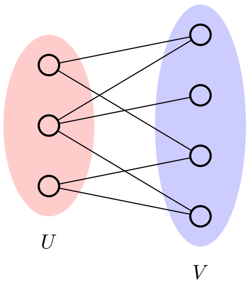

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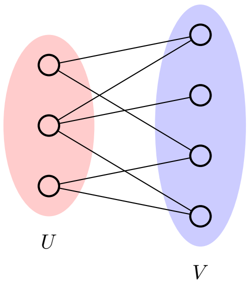

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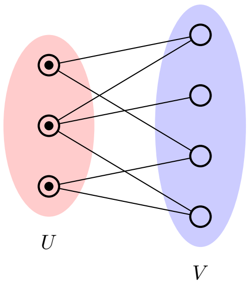

Intuitive observations

- Two fully packed configurations u and v [but possibly many more]

$\longrightarrow$ Both u and v are "locally stable".

$\longrightarrow$ v is the "most efficient" packing.

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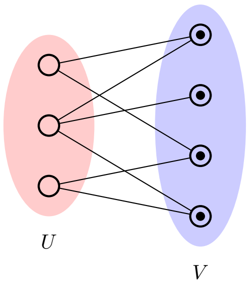

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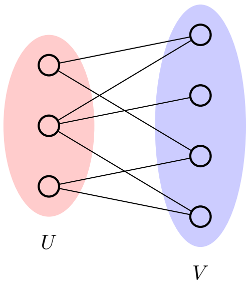

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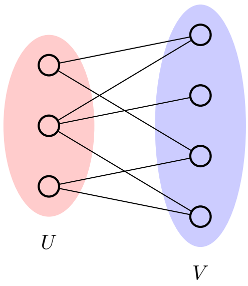

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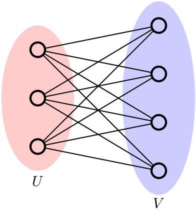
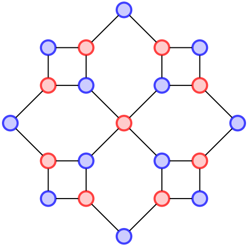
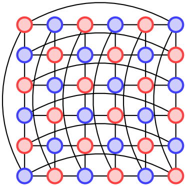
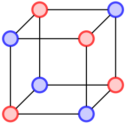
Hard-core process on a bipartite graph

Examples of bipartite graphs









graphs arising from two-species Widom-Rowlinson model

Metastability in Markov processes

Some references

- Kramers (1940)
- large deviations / path-wise approach
  - Freidlin and Wentzell (1960–1970)
  - Cassandro, Galves, Olivieri and Vares (1984–)
  - ...
- potential-theoretic approach
  - Bovier, Eckhoff, Gayrard and Klein (2001–)
  - ...

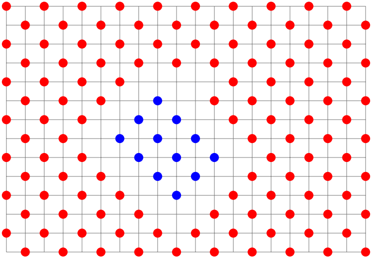
Three books

- Freidlin and Wentzell:
  Random Perturbations of Dynamical Systems (1984)
- Olivieri and Vares: *Large Deviations and Metastability* (2005)
- Bovier and den Hollander:
  Metastability — A Potential-Theoretic Approach (2015)

Main results: I

Theorem (Critical droplets)

For the hard-core dynamics on an even torus $\mathbb{Z}_m \times \mathbb{Z}_n$, when going from u to v, with large probability, the chain passes through exactly one transition $Q \to Q^*$, where Q and Q^* are obtained from the solutions of an isoperimetric problem.



A configuration in Q

[similar for hypercube]

[similar for Widom-Rowlinson]

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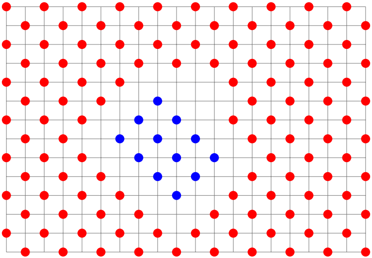


A configuration in Q^*

[similar for hypercube]

[similar for Widom-Rowlinson]

Main results: II

Theorem (Expected transition time)

For the hard-core dynamics on an even torus $\mathbb{Z}_m \times \mathbb{Z}_n$ *we have*

$$\mathbb{E}_u T_v = \frac{\gamma}{2\, m\, n\, l^*} \frac{\lambda^{l^*(l^*+1)+1}}{\bar{\lambda}^{l^*(l^*-1)}} [1 + o(1)]$$

as $\lambda \to \infty$, *where* $l^* := \lceil \frac{1}{\alpha} \rceil$ *is the size of the critical droplet and* $\gamma := |U|\,(1+\lambda) + |V|\,(1+\bar{\lambda})$ *is the rate of the Poisson clock.*

[similar for hypercube]

[similar for Widom-Rowlinson]

Proof steps.

Show that (in discrete time)

$$\mathbb{E}_u T_v = \pi(u)\mathcal{R}(u \leftrightarrow v)\,[1 + o(1)] \qquad \text{as } \lambda \to \infty.$$

Estimate the effective resistance. □

Main results: III

Theorem (Asymptotic exponential law)

For the hard-core dynamics on "many" bipartite graphs we have

[e.g., torus, hypercube, ...]

$$\mathbb{P}_u\left(\frac{T_v}{\mathbb{E}_u\, T_v} > t\right) \to \mathrm{e}^{-t}$$

uniformly in $t \in \mathbb{R}^+$ *as* $\lambda \to \infty$.

Intuition.

Many many trials (attempts to form a critical droplet) with tiny probability of success

$\Longrightarrow$ success time approximately exponential □

Effective resistance: rough estimate

Critical resistance

[a.k.a. communication height]

For every two states $a, b \in \mathscr{X}$, set

$$\Psi(a,b) := \inf_{\omega: a \rightsquigarrow b} \sup_{e \in \omega} r(e)$$

Remark

- *$a, b \mapsto \mathcal{R}(a \leftrightarrow b)$ is a metric on $\mathscr{X}$.*
- *$a, b \mapsto \Psi(a,b)$ is an ultra-metric on $\mathscr{X}$.*

Proposition (Equivalence)

There exists a constant $k \geq 1$ such that

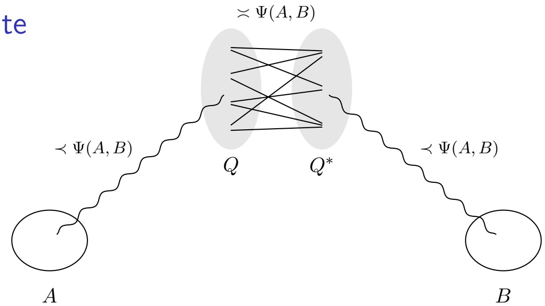
[independent of λ]

$$\frac{1}{k}\,\Psi(a,b) \leq \mathcal{R}(a \leftrightarrow b) \leq k\,\Psi(a,b)$$

for all $a, b \in \mathscr{X}$.

Effective resistance: sharp estimate

Critical gate

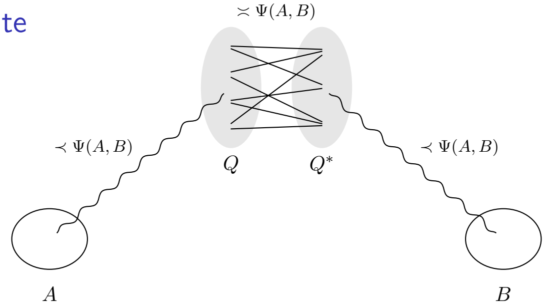


A pair (Q, Q^*) is a critical gate between A and B if

1. $r(x,y) \asymp \Psi(A,B)$ for every $x \in Q$ and $y \in Q^*$ with $x \sim y$,
2. $\Psi(A,x) \prec \Psi(A,B)$ for every $x \in Q$,
3. $\Psi(y,B) \prec \Psi(A,B)$ for every $y \in Q^*$, and
4. every optimal path from A to B passes through a transition $Q \to Q^*$.

Effective resistance: sharp estimate

Critical gate



Proposition

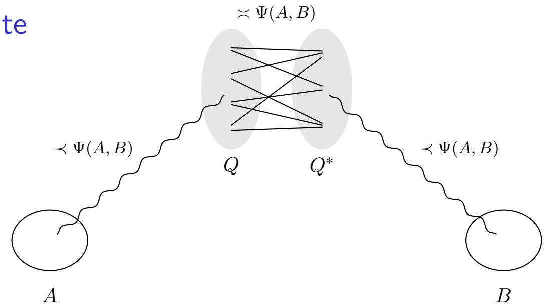
Let (Q, Q^*) be a critical pair between A and B. Then,

$$\mathcal{C}(A \leftrightarrow B) = c(Q, Q^*)\,[1 + o(1)] \qquad \text{as } \lambda \to \infty,$$

where $c(Q, Q^*) := \sum_{x \in Q} \sum_{\substack{y \in Q^* \\ x \sim y}} c(x, y)$.

Effective resistance: sharp estimate

Critical gate



$$\mathcal{C}(A \leftrightarrow B) = c(Q,Q^*)\,[1+o(1)]$$

Proof.

Upper bound: simple Nash-Williams inequality

Lower bound: generalized Nash-Williams inequality

[a.k.a. Berman-Konsowa variational principle]

□

Thank you for your attention!