Restricted density classification in one dimension

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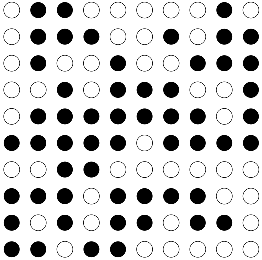
Density classification task



Given: an array of symbols

Task: Determine which symbol is in majority.

Requirements

- A cellular automaton algorithm [say, with periodic boundary]
  [using no extra symbols!]
- Output: must reach consensus on the majority symbol
- Scalability: must work for arrays of arbitrary size

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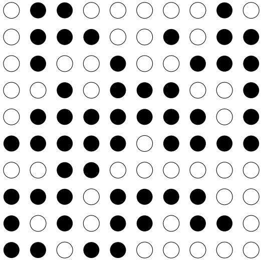

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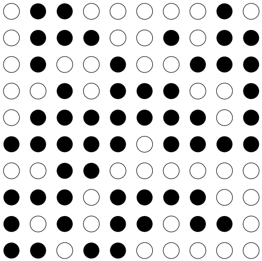

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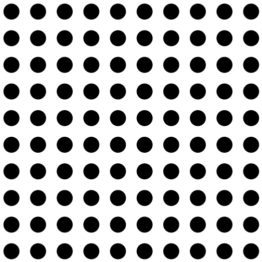

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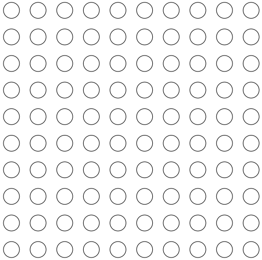

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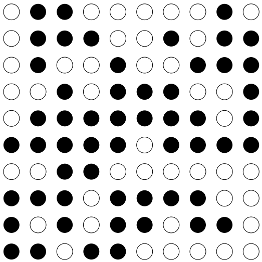

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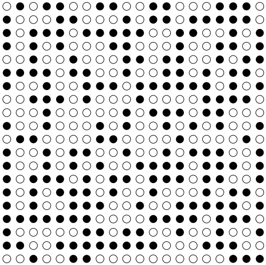

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Unfortunately ... (Land and Belew, 1995)

No perfect solution exists!

Natural relaxation

- classify correctly with "high probability"

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Density classification: infinite lattice

$$\cdots \bullet \circ \bullet \bullet \circ \circ \bullet \circ \circ \bullet \bullet \circ \circ \bullet \circ \bullet \bullet \bullet \circ \bullet \circ \cdots \quad x$$

Terminology

- A configuration of symbols: $x : \mathbb{Z} \to \{\circ, \bullet\}$
- A cellular automaton (CA): $T : \{\circ, \bullet\}^{\mathbb{Z}} \to \{\circ, \bullet\}^{\mathbb{Z}}$
- T classifies x according to density if

$$\begin{aligned} T^t x &\to \texttt{all-}\bullet && \text{if } \texttt{density}_\bullet(x) > 1/2, \\ T^t x &\to \texttt{all-}\circ && \text{if } \texttt{density}_\bullet(x) < 1/2. \end{aligned}$$

as $t \to \infty$. [convergence $\equiv$ site-wise eventual agreement]

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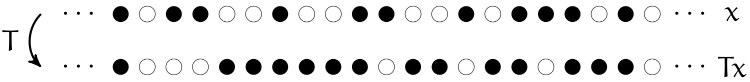

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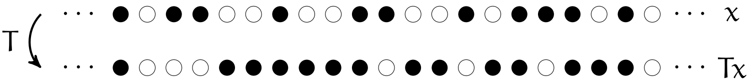

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Task (relaxed)

Classify random configurations with high probability. [Random=?]

Classification of random configurations

$$\cdots \bullet \circ \bullet \bullet \circ \circ \bullet \circ \circ \bullet \bullet \circ \circ \bullet \circ \bullet \bullet \bullet \circ \bullet \circ \cdots \quad X$$

More specifically ...

Let X be a configuration chosen at random, using independent biased coin flips.

$$X_i = \begin{cases} \bullet & \text{with prob. } p, \\ \circ & \text{with prob. } 1-p. \end{cases}$$

Then, $\texttt{density}_\bullet(X) = p$ almost surely. [by the law of large numbers]

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$$\begin{aligned} T^t X &\to \texttt{all-}\bullet && \text{if } p > 1/2, \\ T^t X &\to \texttt{all-}\circ && \text{if } p < 1/2. \end{aligned}$$

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with ~~high probability.~~ probability 1.

Note!

The distribution of X is shift-invariant and ergodic, and $\{T^t X \to \texttt{all-}\bullet\}$ and $\{T^t X \to \texttt{all-}\circ\}$ are shift-invariant events.

Classification of coin-flip configurations

Task (almost-sure classification)

$$T^t X \to \texttt{all-}\bullet \qquad \text{if } p > 1/2,$$
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with probability 1.

Question

Is there a CA that classifies coin-flip configurations almost surely?

[i.e., for any $0 \leq p \leq 1$]

In ≥ 2 dimensions (Bušić, Fatès, Mairesse, Marcovici, 2013)

Perfect solution: Toom's NEC rule!

In 1 dimension

- Not known! But there are candidates . . .
- Modest goal: when the bias is strong . . .

Interpretations & related problems

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- Density classification as recovery from noise
- What if there is noise at every step?
  - Do all-○ and all-● remain stable?

  $\longrightarrow$ In 2d: – Toom's NEC rule (Toom, 1974, 1980)

  $\longrightarrow$ In 1d: – Same candidates ...

  – Gács's sophisticated construction (1986, 2001)
- Density classification as a percolation problem
  - Do contaminations survive (without escaping to infinity)?

  [Contaminations can only spread through neighbours.]
- Sharp phase transition in the behaviour when varying p

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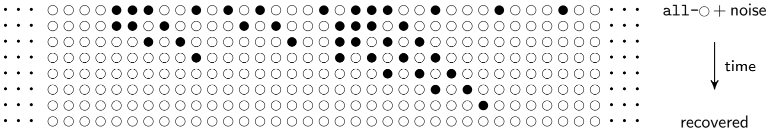

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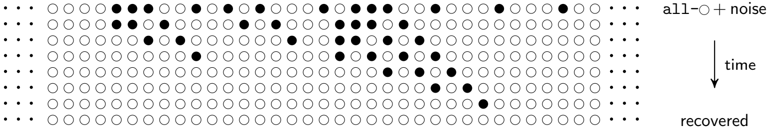

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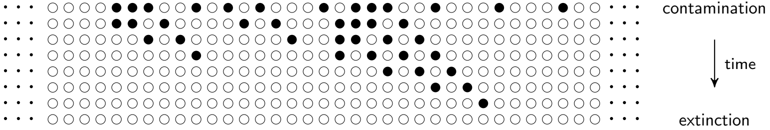

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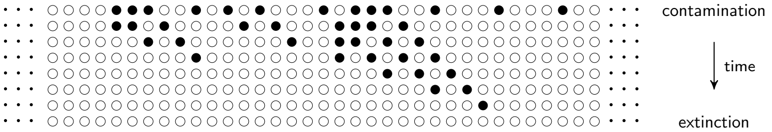

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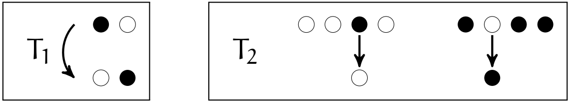

traffic　　　　filter

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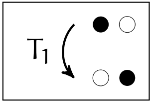

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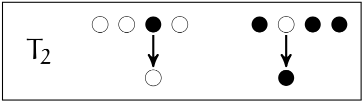

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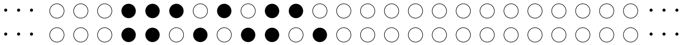

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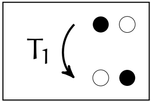

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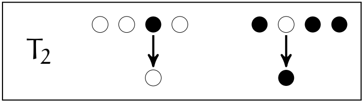

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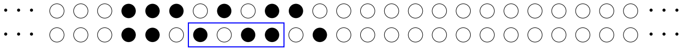

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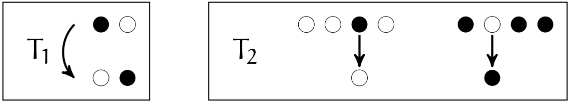

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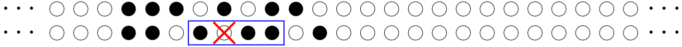

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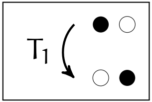

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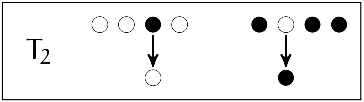

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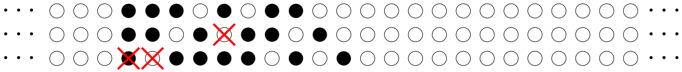

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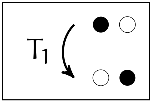

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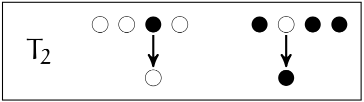

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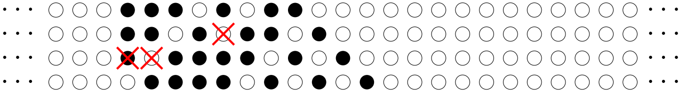

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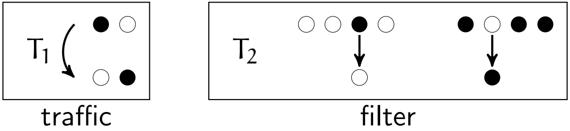

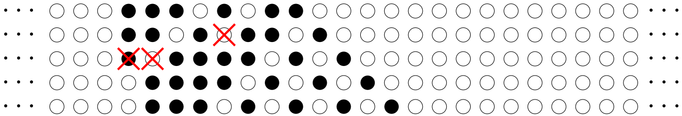

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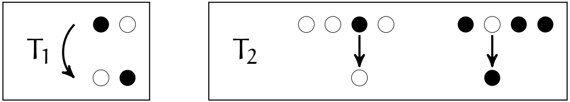

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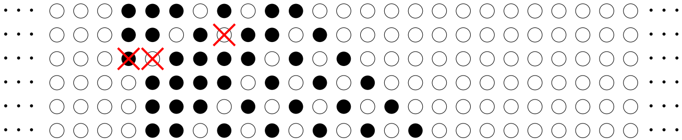

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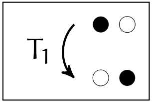

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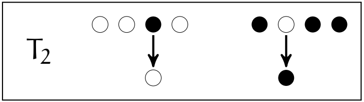

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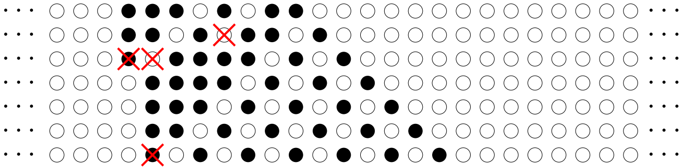

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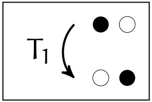

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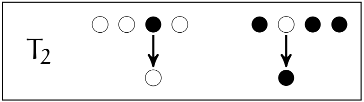

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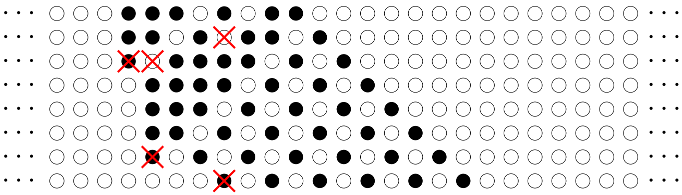

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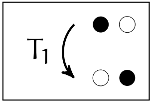
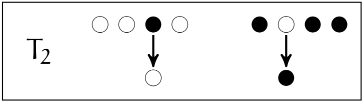
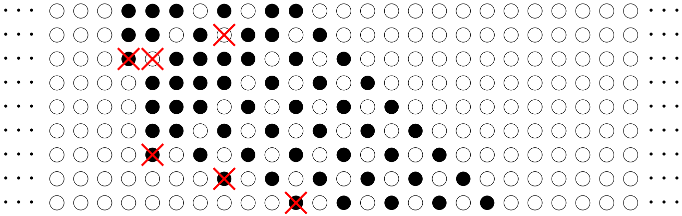


traffic



filter



A finite island of ● in a sea of ○

Symmetry: ○ ↔ ● and `left` ↔ `right`

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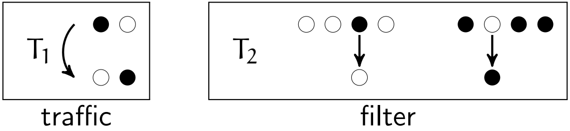

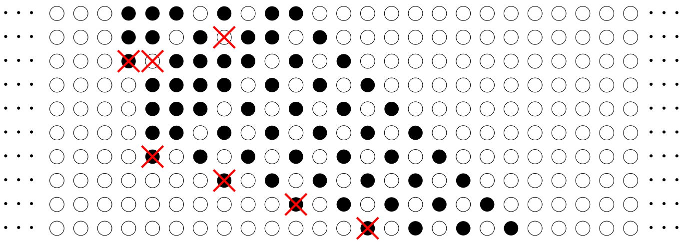

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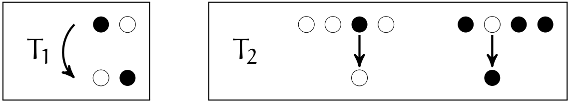
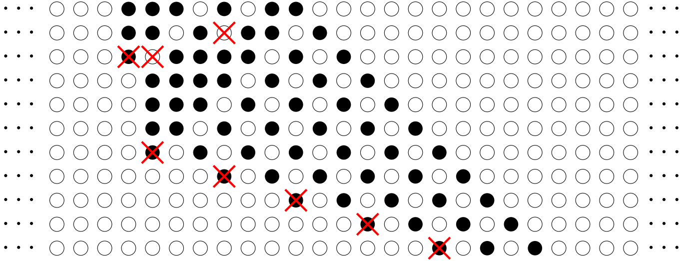


traffic   filter



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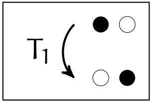

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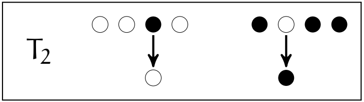

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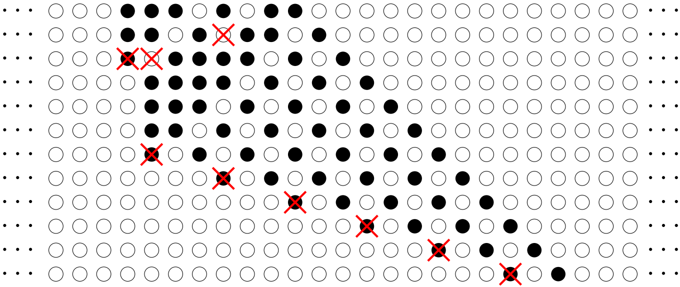

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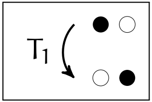

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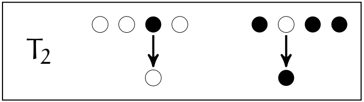

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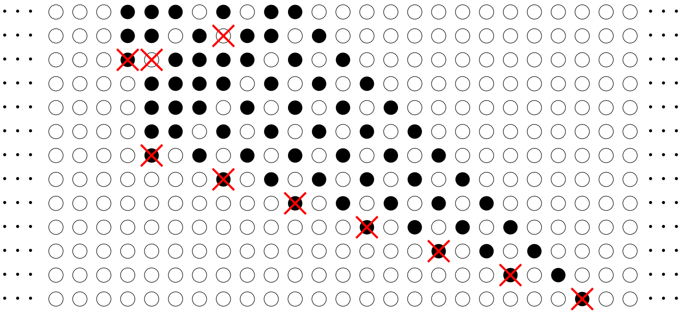

The island is washed out!

Symmetry: $\circ \leftrightarrow \bullet$ and `left` $\leftrightarrow$ `right`

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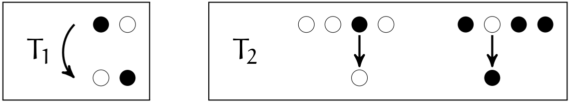

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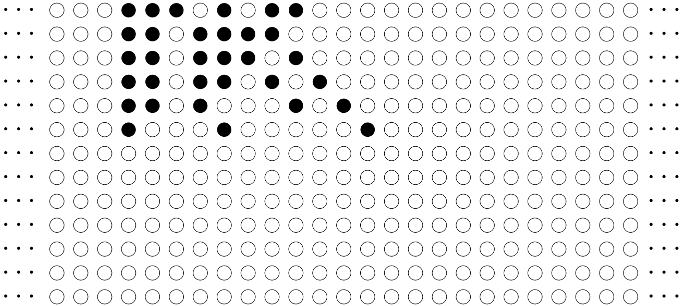
Theorem (Kari and Le Gloannec, 2012)

Every finite island is eventually washed out!
An island of length l is washed out within $2l$ steps.

Symmetry: $\circ \leftrightarrow \bullet$ and `left` $\leftrightarrow$ `right`

Candidate II: GKL (Gács, Kurdyumov, Levin, 1978)

$$(Tx)_i \triangleq \begin{cases} \mathrm{maj}(x_{i-3}, x_{i-1}, x_i) & \text{if } x_i = \circ, \\ \mathrm{maj}(x_i, x_{i+1}, x_{i+3}) & \text{if } x_i = \bullet, \end{cases}$$



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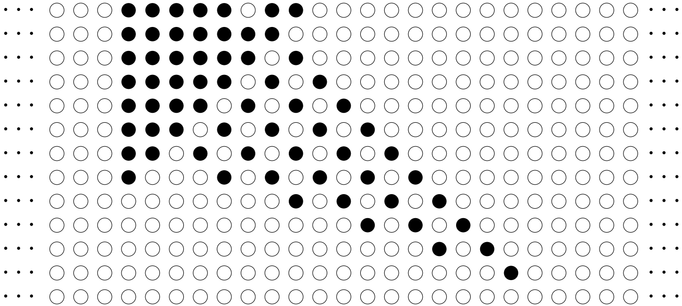

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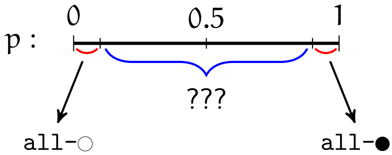
Restricted classification

Theorem (T., 2014)

Let T be modified traffic or GKL. Then, T classified a biased coin flip configuration almost surely correctly provided the bias is strong.



Recall: $p \triangleq \mathbb{P}(X_i = \bullet)$

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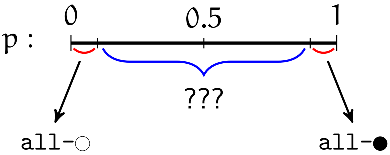

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Remark I

This shows a phase transition when p is varied.
However, it doesn't rule out other phases in between.

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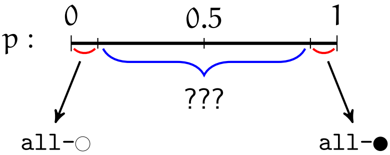
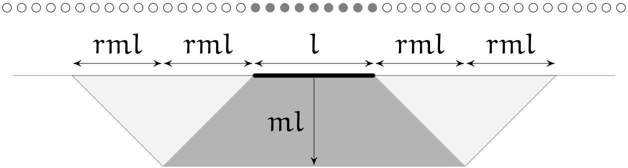


Recall: $p \triangleq \mathbb{P}(X_i = \bullet)$

Remark II

By symmetry, we can focus on p close to 0.

Washing out finite islands (in linear time)



Washing out an island of length l in ml steps

r: neighbourhood radius of the CA

Examples: – modified traffic and GKL with $m = 2$
– (also Toom's NEC rule)

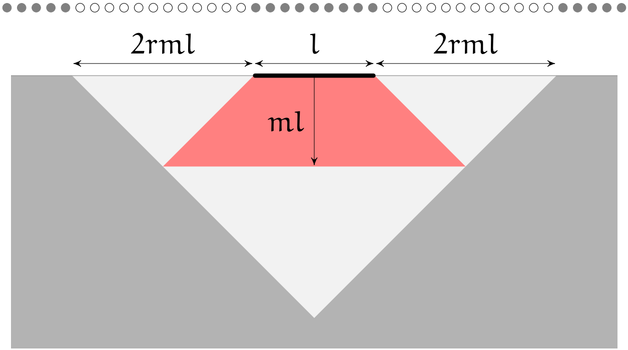
Claim

Suppose T washes out finite islands of ● in background of ○ in linear time. Then,

$$T^t X \to \texttt{all-}\circ \quad \text{almost surely}$$

if X is a coin flip configuration with p close to 0.

Isolated islands



An isolated island has a sufficiently wide margin of ○

Observation

An isolated island disappears before sensing or affecting the rest of the configuration.

$\Rightarrow$ removing an isolated island from a configuration x does not affect whether $T^t x \to \texttt{all-}\bigcirc$ or not.

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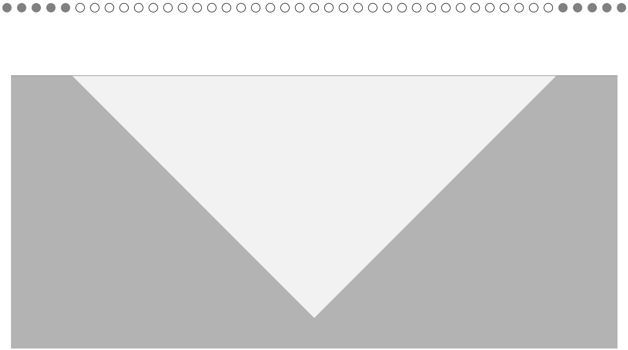

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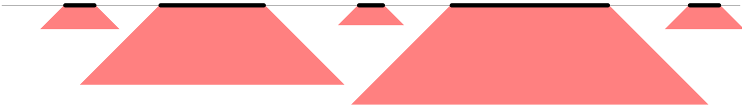
Washing out sparse sets



Cleaning isolated islands makes larger islands isolated!

Cleaning procedure [Gács, 1986, 2001, Durand, Romashchenko, Shen, 2012]

Clean isolated islands recursively.

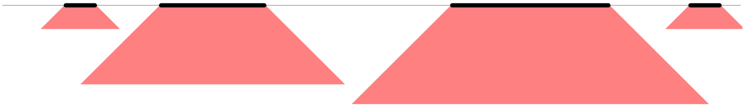
We call a configuration sparse if the cleaning procedure eventually cleans every ●.

$\longrightarrow$ Decompose a sparse configuration x into a family of islands $\mathcal{I}(x)$.

Question

Is sparseness enough for $T^t \to$ `all-`○?

Washing out sparse sets



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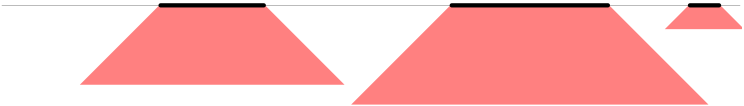

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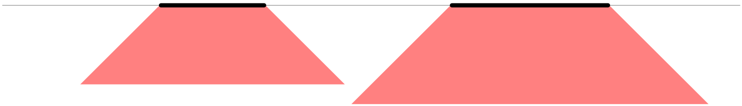

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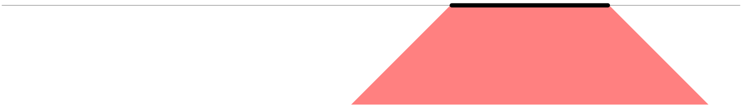

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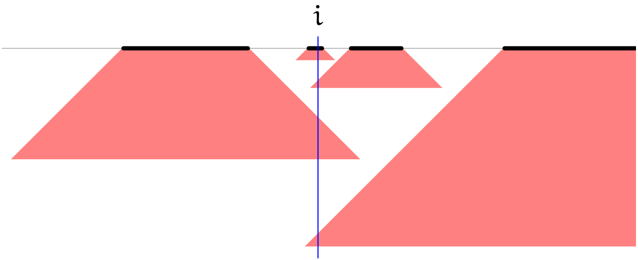


Site i may be changed infinitely often!

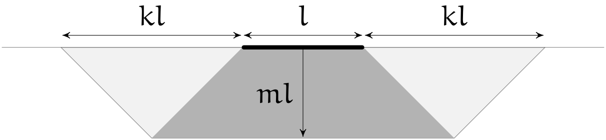
We call a sparse configuration x strongly sparse if every site is in the territory of at most finitely many islands in $\mathcal{I}(x)$.

[territory = island and its margin]

Observation

$T^t x \to$ `all-`$\bigcirc$ if x is strongly sparse.

Sparseness of random configurations



$k \triangleq 2rm$

Theorem (Gács, 1986, 2001, Durand, Romashchenko, Shen, 2012)

A biased coin flip configuration X is almost surely strongly sparse if the parameter p is sufficiently close to 0.
More precisely: it is enough that $p < (2k)^{-2}$

Examples

For modified traffic and GKL, $m = 2$ and $r = 3$, so $k = 12$.

$\Rightarrow$ X is classified almost surely correctly if $p < 0.0017$ or $p > 0.9983$.

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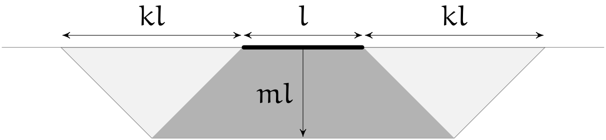

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Choose an appropriate sequence $l_1 < l_2 < l_3 < \cdots$ of lengths.

[to be determined ...]

Cleaning procedure: Version II

1. Clean all isolated islands of length $\leq l_1$.
2. Clean all isolated islands of length $\leq l_2$.
3. ...

Note: The notion of sparseness does not change!

Question

What is the probability that site u has state $\bullet$
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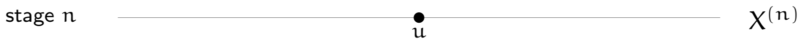

If site u has state ● at stage n, then

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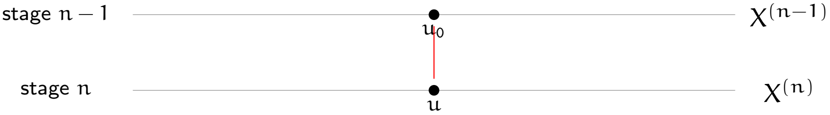

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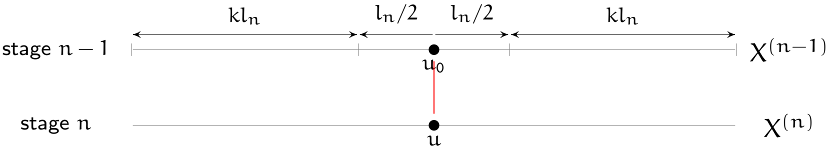

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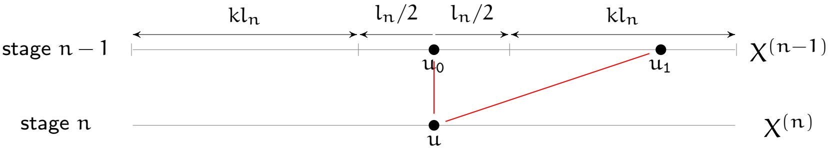

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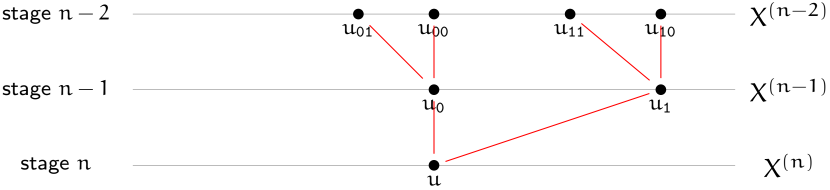

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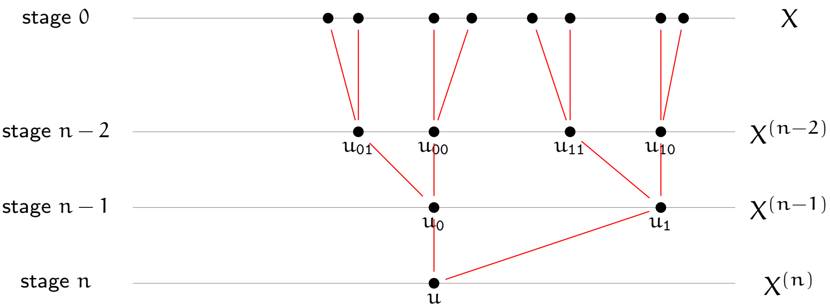

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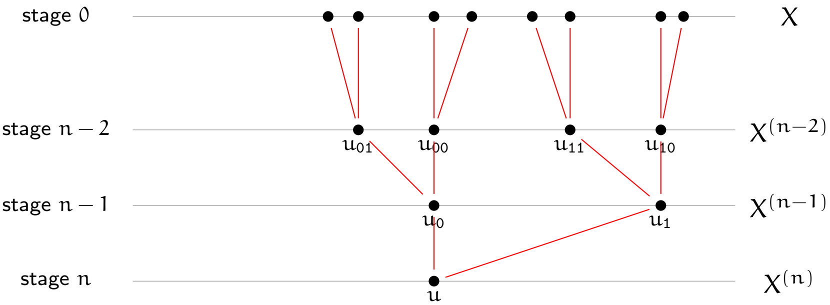


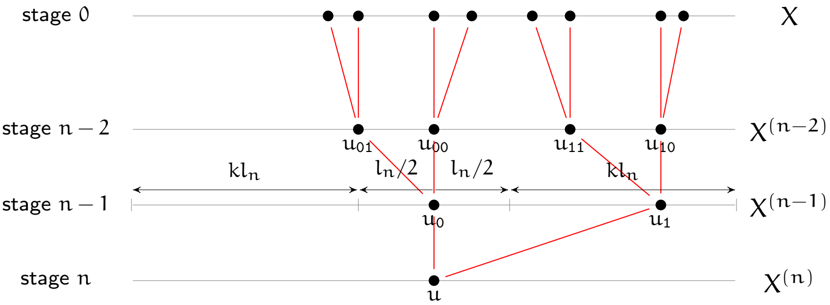
Probability of a tree

Choose $l_n \triangleq (4k+3)^{n-1}$ to sure the leaves are distinct!

$$\mathbb{P}(\text{a given tree is an explanation}) = p^{2^n}$$

Sparseness of random configurations: proof

Explanation tree



Number of a tree

Recursive inequality: $f_n \leq 2kl_n \times f_{n-1}^2$.

$\#$ of trees $\leq (2k)^{2^{n+1}}$

Sparseness of random configurations: proof

Probability of a tree

$$\mathbb{P}(\text{a given tree is an explanation}) = p^{2^n}$$

Number of a tree

$$\# \text{ of trees} \leq (2k)^{2^{n+1}}$$

Probability of survival after n cleaning steps

At least one explanation tree must exist!

$$\mathbb{P}(X_u^{(n)} = \bullet) \leq (2k)^{2^{n+1}} p^{2^n} = \left((2k)^2 p\right)^{2^n}$$

which $\to 0$ as long as $p < (2k)^{-2}$.

Sparseness of random configurations: proof

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$$\mathbb{P}(X_u^{(n)} = \bullet) \le \left((2k)^2 p\right)^{2^n}$$

which $\to 0$ as long as $p < (2k)^{-2}$.

Corollary: sparseness

X is sparse as long as $p < (2k)^{-2}$.

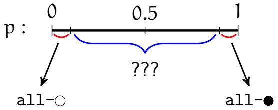
Corollary: strong sparseness

X is strongly sparse as long as $p < (2k)^{-2}$. [by Borel-Cantelli ...]

Q.E.D.

Remarks and open problems

- Does an intermediate phase exist?



- Random configurations with other distributions
  Sparseness? Classification? [Markov, Gibbs, . . .]
- Would sparseness approach work for probabilistic CA?
  - Fatès, 2013: majority-traffic
  - Noisy version of majority
- Noise at each step: modified traffic and GKL
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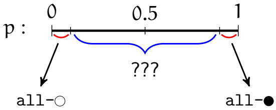

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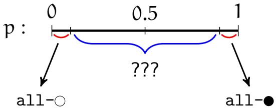

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  Sparseness? Classification? [Markov, Gibbs, ...]
- Would sparseness approach work for probabilistic CA?
  - Fatès, 2013: majority-traffic
  - Noisy version of majority
- Noise at each step: modified traffic and GKL
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  - Törmä: one-dimensional SFT [in progress]
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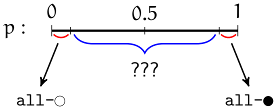

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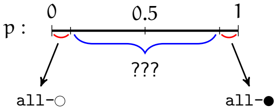

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Thank you for your attention!