COMBINATORICS OF CONSERVATION LAWS

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Abstract

Conservation laws in physics are numerical invariants of the dynamics of a system. This article concerns conservation laws in a fictitious universe of a cellular automaton. We give an overview of the subject, with particular attention to problems of combinatorial flavor.

1 Kepler's Laws and Selective Observation

Much of science arises from looking at nature through a highly selective glass, which eliminates all the irrelevant details and singles out a particular feature to be studied. The strength of this approach flourishes when the filtered feature has a nice description of its own, which is independent of the eliminated parts. A conservation law is the simplest of such descriptions. It asserts that a certain quantity associated to a system remains constant throughout the evolution of the system.

Perhaps the first example of a conservation law found in physics is the intriguing discovery by the German astronomer Johannes Kepler (1571–1630) [27] of the laws governing the motion of the planets. Kepler knew from his large collection of data, gathered from astronomical observations, that each planet follows, not a perfectly circular orbit, but an elliptic one, with sun on one of the focal points. The speed of the planet is not uniform either. Whenever it is farther from the sun, the planet moves slowly, while once it comes closer to the sun it circles around

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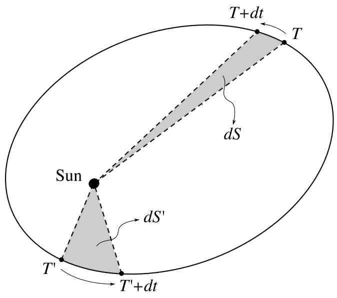


Figure 1: Kepler's selective observation: $dS' = dS$

it faster. Kepler was able to put this quantitatively, by realizing that the axis connecting the planet to the sun sweeps out equal areas within equal time segments (Figure 1). In other word, all through its orbit, the area-sweeping rate dS/dt of each planet remains constant (cf. [1]).[1]

In this article we discuss such laws in cellular automata. A *cellular automaton* (CA for short) is an abstract structure, consisting of a d-dimensional checkerboard ($d = 1, 2, 3, \ldots$). Each cell of the board has a state chosen from a finite set of states. The state of each cell changes with time, according to a uniform, deterministic rule, which takes into account the previous state of the cell itself and those in its neighborhood. The changes, however, happen synchronously, and in discrete time steps.

One of the simplest CA exhibiting a non-trivial conservation law is the *Traffic* CA, which resembles cars moving on a highway. This is a one-dimensional CA, consisting of an infinite number of cells arranged next to each other on a line. Each cell has two possible states: ■ (interpreted as a "car") or □ ("empty space"). At each step, a car moves one cell forward, if and only if, its front cell is empty. Figure 2 shows a typical space-time diagram of the evolution of the Traffic CA. Not surprisingly, the number of cars on the highway is preserved by the evolution of the CA.

As a two-dimensional example, consider the following discrete model of an excitable medium due to Greenberg and Hastings [16]. The CA runs on a two-dimensional board. Each cell is either "at rest" (state □), or "excited" (state ■), or is in a "refractory phase" (state ▩). A cell which is at rest remains so unless it is "stimulated" by one or more of its four neighbors (i.e., if at least one of the neighbors is excited). An excited cell undergoes a 1-step refractory phase, before going back to rest. Typically, a configuration of the infinite board contains a number of "singularities" with waves continuously swirling around them. See Figure 3

[1] This was later coined the law of conservation of angular momentum.

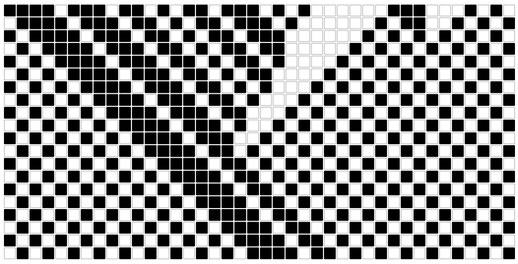
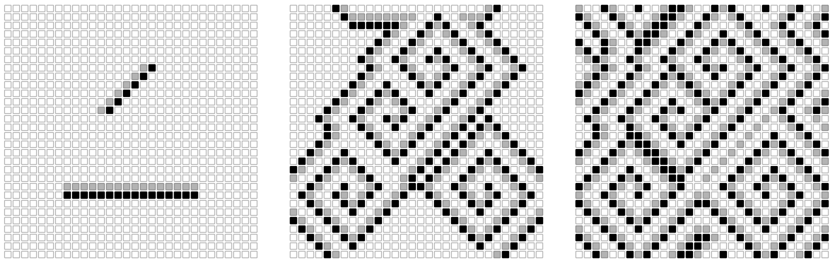


Figure 2: A typical space-time diagram of the Traffic CA. Time evolves downward. The highway is directed toward left.



(a) (b) (c)

Figure 3: Simulation of Greenberg-Hastings model on a spatially periodic configuration. (a) The initial configuration. (b) The configuration at time $t = 10$. (c) The configuration at time $t = 60$.

for a few snapshots. The singularities are never created, nor are they destroyed. Therefore, the number of such singularities remains constant throughout time. To put it precisely, the singularities are the 2×2 blocks of cells with states , ,  or their rotations or mirror images. It is a matter of mechanical verification to see that a singular 2×2 block remains singular after one step, and a non-singular one remains non-singular. See [16, 15, 17] for the fascinating study of this CA and the like.

2 Conservation Laws: How to verify them?

The first thing we may want to know is whether we can algorithmically verify the validity of a certain conservation law in a cellular automaton. But first we need to fix the notations and make it clear what exactly we mean by a conservation law.

The cells of a d-dimensional checkerboard are indexed by the elements of $\mathbb{Z}^d$. The *state set* of the CA is a finite set S. By a *configuration* of the board we mean any mapping $c : \mathbb{Z}^d \to S$ that assigns a state to each cell on the board. When $A \subseteq \mathbb{Z}^d$ is finite, an assignment $p : A \to S$ is called a *pattern*. The *neighborhood* is specified by a finite set $N \subseteq \mathbb{Z}^d$. The neighborhood of a cell $i \in \mathbb{Z}^d$ is the set $i + N \triangleq \{i + k : k \in N\}$. The neighborhood of a set A of cells is the set $A + N \triangleq \{i + k : i \in A, k \in N\}$. For convenience we always assume $0 \in N$. The state of the cells are updated by a *local rule*, which is a function $f : S^N \to S$ that assigns a new state $f(p)$ to each neighborhood pattern $p : N \to S$. The *global mapping* of the CA is the mapping $F : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$, that maps each configuration c of the board to a new configuration $F(c)$, in which a cell i has state $F(c)[i] \triangleq f(c[i + N])$.[2] The evolution of the CA starting from an initial configuration c is seen as the iteration of F on c.

Two configurations c and e are said to be *asymptotic*, if they agree on all but possibly finitely many cells; i.e., if the set $\{i : c[i] \neq e[i]\}$ is finite. When s is an arbitrary state, the *s-uniform* configuration, denoted by c_s, is the one with s on every cell. An *s-finite* configuration is one which is asymptotic to the s-uniform one. A state s is called *quiescent*, provided $F(c_s) = c_s$. If s is quiescent, F maps s-finite configurations to s-finite ones.

Suppose that each state $s \in S$ is given a real number $\mu(s)$ which we call the *energy* of that state. The energy-content of a pattern $p : A \to S$ is the sum $\mu(p) \triangleq \sum_{i \in A} \mu(p[i])$. To assert that the energy μ is "conserved" by the CA, we might be tempted to require that the μ-content of every configuration c is preserved by the application of F. However, the μ-content of a configuration in general is not well-defined, as there are infinitely many cells on the board. Here there are several approaches one can take, that are all more or less equivalent. See [19, 8, 28, 10] for a handful.

When two configurations c and e are asymptotic, the *difference* between their μ-content can be defined as

$$\delta\mu(c, e) \triangleq \sum_{i \in \mathbb{Z}^d} [\mu(e[i]) - \mu(c[i])] \tag{1}$$

(only finitely many terms are non-zero). We say that F *conserves* μ, if

$$\delta\mu(F(c), F(e)) = \delta\mu(c, e) , \tag{2}$$

for every two asymptotic configurations c and e (Figure 4).

[2] Here we write $g[A]$ for the restriction of the mapping g to the subset A of its domain.

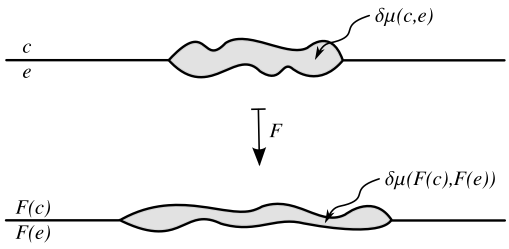


Figure 4: Conservation law: $\delta\mu\,(F(c), F(e)) = \delta\mu\,(c, e)$

The car conservation law in the Traffic CA comes with the energy valuations $\mu(\blacksquare) \triangleq 1$ and $\mu(\square) \triangleq 0$. To express the conservation of the number of singularities in the Greenberg-Hastings model we need energy functions that assign energy to patterns, instead of single states. We shall discuss that in Section 5.

The conservation of the number of cars in the Traffic CA could also be stated in a different way, by saying that whenever there are only finite number of cars on the highway, their number is preserved throughout time. In general, if $\mu(\diamond) = 0$ for a quiescent state $\diamond \in S$, we can define the μ-content of every $\diamond$-finite configuration c by the sum $\mu(c) \triangleq \sum_{i\in\mathbb{Z}^d} \mu\,(c[i]))$ with no problem. This gives rise to an equivalent definition of a conservation law:

Proposition 1. *Let $\mu : S \to \mathbb{R}$ be an energy assignment, and suppose that $\mu(\diamond) = 0$ for a quiescent state $\diamond \in S$. The following are equivalent:*

a) $\mu\,(F(c)) = \mu\,(c)$, for every $\diamond$-finite configuration c.

b) $\delta\mu\,(F(c), F(e)) = \delta\mu\,(c, e)$, for every two asymptotic configurations c and e.

Is there an easy way to verify the validity of an arbitrary conservation law in a cellular automaton? Neither of the two definitions above is directly helpful, because they involve infinitely many equalities. Playing a bit with the Traffic CA we may convince ourselves that it indeed conserves the number of cars: An empty highway clearly remains empty. If we add or remove a single car from a configuration only 3 cells may notice the change in one step (the cell where we put the car, and its left and right neighbors). Therefore, we can verify that such a change does not destroy the validity of the conservation law. In fact, this is all we need to do! Notice that starting from the empty highway we can reach any finite configuration of the highway by adding cars on it one by one.

The same idea works in general. The key observation is that for every two asymptotic configurations c and e, there is a finite sequence $c = c_0, c_1, \ldots, c_n = e$, such that c_i and c_{i-1} differ on exactly one cell. Furthermore,

$$\delta\mu(x, z) = \delta\mu(x, y) + \delta\mu(y, z) \tag{3}$$

whenever x, y and z are asymptotic configurations. Therefore, to verify that an energy μ is conserved by a CA F, one needs to verify Equation (2) only for those configurations c and e that differ on exactly one cell.

Proposition 2 (Hattori and Takesue [19])**.** *An energy μ is conserved by a CA F, if and only if, Equation (2) holds for every c and e that agree everywhere except on a single cell.*

Proposition 2 immediately gives an algorithms that verifies whether given a conservation law is valid within a given cellular automaton. In fact, it allows us to find *all* such conservation laws held in the CA. The set of energy assignments $\mu : S \to \mathbb{R}$ that are conserved by a CA F form a vector space — the solution space of a finite system of linear equations obtained from (2) for every c and e that differ on exactly one cell — which can be found effectively.

3 Flow of Energy and Local Conservation Laws

A conservation law, as we defined it above, is a global property. It asserts that the energy is globally preserved, but it does not explain how this energy is redistributed on the configuration at each step. Is this redistribution local? Or are cells arbitrarily far from each other working together (whatever it means) to keep the total amount of energy intact? The answer to the latter question is intuitively negative, but we would like to be able to express the microscopic dynamics of the energy in terms of "flows" of energy to understand things better.

More specifically, on every configuration x, we would like to assign a value $\Phi_{i\to j}(x)$ to each pair of cells i and j (not too far from each other), as the amount of energy *flowing* from i to j, such that the in-coming and out-going flows of each cell are compatible with the energy of that cell (Figure 5(a)). Furthermore, this value should depend only on the states of a limited number of cells in the vicinity of i and j.

To be precise, a *flow* for an energy μ is a mapping $x, i, j \mapsto \Phi_{i\to j}(x) \in \mathbb{R}$ that satisfies the following conditions:

a) For every configuration x and every cell a,

$$\mu\left(x[a]\right) = \sum_{j\in\mathbb{Z}^d} \Phi_{a\to j}(x)\,, \tag{4}$$

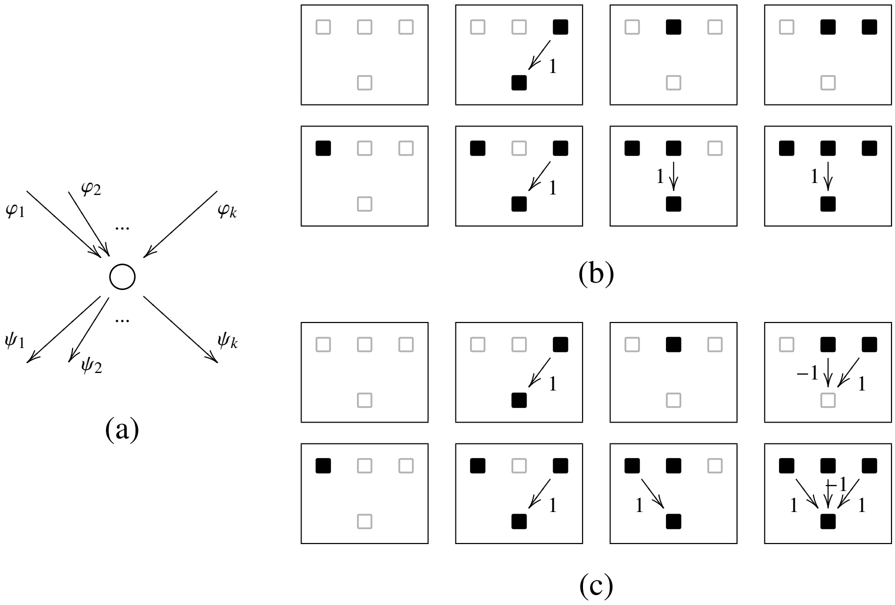


Figure 5: (a) Continuity of the flow: $\sum_i \varphi_i = \mu = \sum_j \psi_j$.
(b, c) Two different flows for the car conservation law in the Traffic CA.

b) For every configuration x and every cell a,

$$\sum_{i \in \mathbb{Z}^d} \Phi_{i \to a}(x) = \mu\left((Fx)[a]\right) , \tag{5}$$

c) There exist finite sets $K, I \subseteq \mathbb{Z}^d$, and a rule $\varphi : S^K \times I \to \mathbb{R}$ such that,

$$\Phi_{i \to j}(x) = \begin{cases} \varphi\left(x[j+K], i-j\right) & \text{if } i-j \in I, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

for every $x \in S^{\mathbb{Z}^d}$ and $i, j \in \mathbb{Z}^d$.

Equations (4) and (5) are called the *continuity equations*. Equation (6) states that the amount of the flows toward each cell is decided locally, by looking at a finite *neighborhood* K of that cell. The set I is the set of *directions* from which energy flows into a cell.

An energy μ is *locally conserved* by F, if it has a flow. In cellular automata, conservation laws and local conservation laws (as we defined them) are equivalent concepts.

Proposition 3 (Hattori and Takesue [19]). *In cellular automata, conserved energies are locally conserved.*

The argument is a refinement of that in Proposition 2. Let us emphasize that the choice of the flow mapping Φ is not unique. Figures 5(b) and 5(c) show two different flows for the conservation of cars in the Traffic CA. In fact, every conservation law has infinitely many flows compatible with it. See Sections 4 and 6 below.

4 Pebbles, Matchings and the Quanta of Energy

Let us introduce a game which is, not quite untypically [2], played by the cells of a cellular automaton.[3] We start with an arbitrary configuration x. Initially, each cell on the board is given a number of pebbles, depending only on its current state. That is, every cell i is given $\eta(x[i])$ pebbles, where $\eta : S \to \mathbb{Z}^{\geq 0}$ is arbitrary. At each step, every cell (besides changing its own state) is to distribute its pebbles among a number of cells in its vicinity (it can keep some for itself), in such a way that after one step, each cell has exactly as many pebbles as is assigned to its new state. That is, after one step, cell i must have $\eta(F(x)[i])$ pebbles. Is there a uniform and local strategy for each cell to win the game on every configuration? By "uniform" we mean that every cell should use the same strategy which is independent of the configuration of the board. By "local" we mean that every cell is only allowed to look at the states of a bounded number of cells around it, to make its decision.

You might have recognized that such a strategy is nothing but an energy flow with values from non-negative integers. Therefore, we immediately see that a necessary condition for the existence of a winning strategy is that the CA conserves η. But is it also sufficient? In other words, given a conserved energy $\eta : S \to \mathbb{Z}^{\geq 0}$, is it always possible to construct a suitable *pebble redistribution* rule (i.e., a flow with non-negative integer values)? Such a rule would provide us with a clear understanding of the dynamics of η in terms of the local movements of the tiniest bits of η — the "quanta" of η.

Proposition 4 (Fukś [14] and Pivato [28])**.** *For every one-dimensional CA F, and every conserved pebble assignment η, the pebble redistribution game has a winning strategy.*

The existence of a winning strategy in higher-dimensional CA is still open. See [21] for a partial solution in two dimensions. This game and its various generalizations are addressed in [23, 26, 7, 3].

Rather than discussing the argument behind Proposition 4, let us point out a natural representation of a pebble redistribution rule (in any number of dimensions) as a perfect matching in a bipartite graph. A *matching* in a graph $G = (V, E)$,

[3] Here, however, unlike in the Conway's game, there is something to win!

is a collection $M \subseteq E$ of edges such that no two of them are incident. A matching M *covers* a subset $A \subset V$ of vertices, if every vertex in A is incident to an element of M. A *perfect* matching is one which covers all the vertices.

Let $\eta : S \to \mathbb{Z}^{\geq 0}$ be a pebble assignment function. Given a configuration x, let us construct a bipartite graph $G[\eta, x] = (U, V, E)$ in the following way. For each pebble on x, the graph has a vertex which is in U. Similarly, for every pebble on $F(x)$, there is vertex inside V. A pebble $u \in U$, coming from a cell i, is connected by an edge to a pebble $v \in V$, coming from a cell j, if and only if i is a neighbor of j (i.e., if and only if $i - j \in N$, N being the neighborhood of the CA).

A perfect matching in graph $G[\eta, x]$ can be interpreted as a way of moving the pebbles on configuration x, so that we obtain a redistribution of pebbles as on configuration $F(x)$. In particular, an edge between a pebble $u \in U$ on cell i and a pebble $v \in V$ on cell j would mean that u is moved from cell i to cell j.

A necessary and sufficient condition for the existence of a perfect matching in a bipartite graph is given by the so-called marriage theorem due to Philip Hall (see e.g. [33]). The marriage theorem states that a (possibly infinite, but locally finite[4]) bipartite graph $G = (U, V, E)$ has a matching that covers U, if and only if, for every finite set $A \subset U$, the number of vertices in V that are adjacent to A is at least $|A|$. The latter is called the Hall's marriage condition.

It is not difficult to see that whenever η is conserved by F, the graph $G[\eta, x]$ associated to every configuration x satisfies the Hall's marriage condition. As a result we obtain the following.

Proposition 5 (Pivato [28])**.** *An energy $\eta : S \to \mathbb{Z}^{\geq 0}$ is conserved by the CA F, if and only if, for every configuration x, the graph $G[\eta, x]$ has a perfect matching.*

Notice that a perfect matching in $G[\eta, x]$ provides a mapping $i, j \to \Phi_{i \to j}(x) \in \mathbb{Z}^{\geq 0}$ that satisfies the continuity equations (Equations (4) and (5)), and which is zero whenever $i \notin j+N$. It remains open whether such a mapping can be generated by a local rule, as in Equation (6).

5 Energy of Interaction

So far we have discussed energy functions that assign a value to each single cell, independent of its context. In a more general setting, we could associate energies to each particular pattern that a number of cells close to each other would make. We have already seen an example of a conservation law with such an energy in Section 1. Another example is the conservation of energy in the Ising model (see e.g. [22, 34, 31, 9]).

[4]A graph is *locally finite* if every vertex has a finite degree.

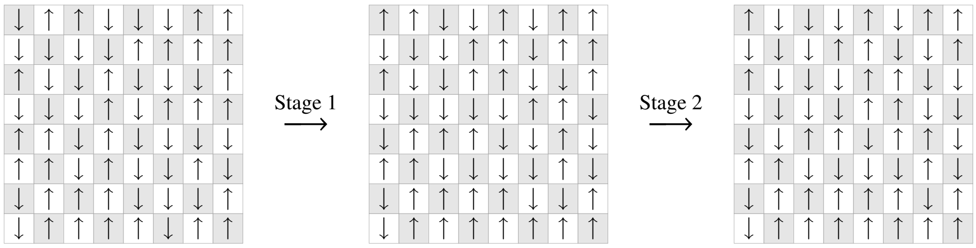


Figure 6: The two stages of updating a random spatially periodic configuration in the two-dimensional deterministic Ising model.

The Ising model[5] tries to capture the behavior of ferromagnetic materials in an abstract setting. Each cell on the board represents a spin, directing upward (state $\uparrow$) or downward (state $\downarrow$). A pair of adjacent spins has an energy attached to it. Aligned pairs have energy -1, while anti-aligned ones have energy 1. In other words, every pattern $p : \{a, b\} \to \{\uparrow, \downarrow\}$ in which $a, b \in \mathbb{Z}^2$ are adjacent cells has energy

$$\mu(p) \triangleq -\varsigma(p[a]) \cdot \varsigma(p[b]) , \tag{7}$$

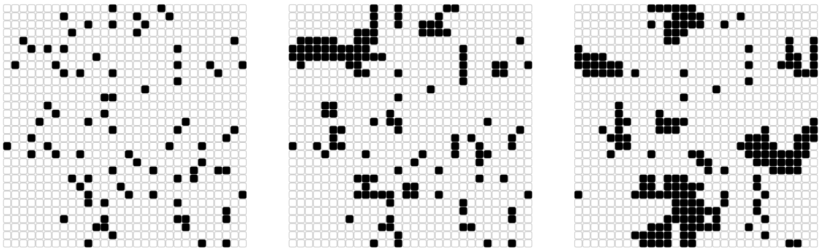
where $\varsigma(\uparrow) = 1$ and $\varsigma(\downarrow) = -1$.

In the deterministic Ising model, the CA is updated in two stages. Imagine that the cells are painted black and white, as on the chess board. At the first stage, all the black cells are updated in the following way: A spin on a black cell is flipped (from $\uparrow$ to $\downarrow$, or from $\downarrow$ to $\uparrow$), if and only if the change does not make any difference in the total value of energy in the neighborhood of the cell. This is the case, when the number of upward spins adjacent to the cell is the same as the the number of downward spins. At the second stage, the white cells are updated in a similar fashion (Figure 6). It is clear that the energy μ is conserved by this CA.[6] Figure 7 shows a few snapshots from a simulation of the Ising CA.

A *higher-range* energy is obtained by assigning values to a finite number of different patterns. We leave it to an interested reader to work out the precise formalism. Conservation of such an energy can be defined in a similar fashion. Variations of Propositions 1, 2 and 3 can be obtained likewise. In particular, in

[5] Named after the physicist Ernst Ising (1900–1998).

[6] Strictly speaking, according to our definition, the above model is not a cellular automaton, as the local updating rule depends on the position of the cell (i.e., whether it is black or white). However, there are various ways to make a CA out of it. For example, we could choose the 2×2 blocks of the board as the cells of the CA. Or we could weave two independent configurations of the board together and run the model on both simultaneously. See [32] for more details.



(a) (b) (c)

Figure 7: Simulation of the Ising model on a spatially periodic configuration. Black represents ↑. White represents ↓. (a) The initial configuration. (b) The configuration at time $t = 10$. (c) The configuration at time $t = 60$.

every CA the conserved energies of a certain range form a linear space which can be identified effectively. Consult [19, 20] for the details.

Naturally, we would like to have all conservation laws (of arbitrary range) that are valid within a CA. Is there an effective method to find all conservation laws for a given CA? One can of course enumerate the conservation laws of a CA one by one, by solving the above-mentioned linear equations associated to wider and wider ranges, but we may hope for a better, more compact way of presenting them all at once. More specifically, how can we determine whether a given CA has any conservation law at all?

It turns out that no effective way could exist to say whether a given CA has or has not any (non-trivial) conservation law. In two- and higher-dimensional CA, an easy argument can be obtained from the undecidability of finite tiling problem [20]. In one-dimensional CA the undecidability of this question has its root in the existence of very long transients (see [12]).

As a final example, let us present a non-trivial one-dimensional CA with a variety of conservation laws of arbitrary range. This is a simple symmetric two-state CA which exhibits characteristics similar to those of solitons, and was discovered by Bobenko and others [5]. *Solitons* are solitary packets of wave, that travel steadily with constant speed. Upon collision they pass through each other unchanged, with only a shift on their phase.

To maintain its symmetry, the CA is visualized on a diagonally oriented checkerboard (Figure 8). To update its configuration, the CA uses not only the current state of the cells, but also their state one step before.[7] The local rule is

[7] Again, according to our convention, this is not precisely a cellular automaton, but can be

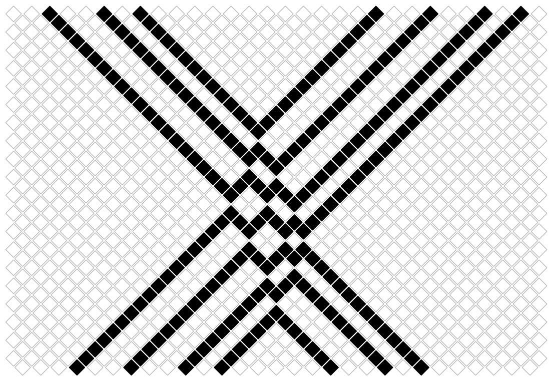
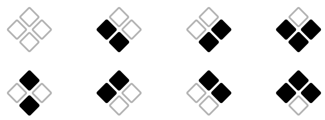


Figure 8: Solitons pass through each other with only a phase shift.

given by



(8)

and is interpreted this way: The left and the right corners of each square are the states of two adjacent cells in the current configuration. The upper and the lower corners depict the states of the cell in between, one step before, and one step after. The left, the right, and the upper states are used to obtain the lower state. Two soliton-like species exist in this CA: one traveling to the right, and one traveling to the left (Figure 8). The sequence of signals traveling to the left and to the right are eventually intact, even though upon meeting each other, they experience momentary perturbation. Therefore, any particular finite pattern of signals traveling to the left (or to the right) gives rises to a conservation law — that the total number of such patterns is conserved. The exact formulation of the underlying energy might be rather technical, but the logic is simple.

6 Open Problems for Curious Souls

Does the pebble redistribution game (see Section 4) always have a winning strategy in two and higher dimensions? In other words, can the conservation of every $\mathbb{Z}^{\geq 0}$-valued energy in a two- or higher-dimensional CA be described by a $\mathbb{Z}^{\geq 0}$-valued flow?

The answer is most likely positive. Nonetheless, even if that is the case, there will be an infinite number of such pebble flows. Is there a (possibly more re-

easily turned into one [32].

stricted) concept of "flow" which is the most natural, in the sense that, it is *unique* and intuitively plausible? One criterion for naturalness is that for a reversible CA (i.e., one in which the time evolution can be traced backward by another CA), the flows in the backward direction of time should be obtained from the flows in the forward direction, only by reversing the direction of the arrows.

Inspiring from the physicists, one approach would be to look for a kind of "variational principle" which distinguishes a single natural flow. A *variational principle* is a way of explaining the particular behavior of a system, by asserting that the system seems to minimize a certain function (cf. [30]). A nice example is the Fermat's principle[8] in optics, which explains the phenomena of reflection and refraction, by postulating that, when traveling from one point to another, light always "chooses" the path that takes the least time. For the one-dimensional pebble redistribution, such a principle indeed exists, and states that the pebbles, in total, should stay as close to their initial positions as possible. Such a pebble flow is always unique. It is the one that preserves the order of pebbles [26].

Given the importance of the *reversible* cellular automata (cf. [31, 32]), it is worth trying to understand the possible mechanisms of energy conservation in this particular class of CA. Among the examples in this article, the Ising CA and the soliton CA are reversible. It is not known whether every reversible CA has a non-trivial conservation law or not. If that is not the case, we may wonder whether there is an effective way to determine the existence of a non-trivial conservation law for this subclass CA (see [12]).

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